Matrix Lie Groups

Thang Pang Ern

This note explains why unitary and orthogonal groups, together with their special variants, are central to Lie Theory[1]. We build from Linear Algebra you might already know (inner products, eigenvalues, rotations) to core ideas in Lie Theory such as Lie algebras, the exponential map exp, maximal tori, and Weyl groups. We also emphasise that these matrix groups are the *canonical compact* models of Lie Theory. In fact, Lie Theory is a course offered at NUS and is coded MA5211.

I gained some interest in this topic in July 2025 when I was doing a UROPS (which stands for Undergraduate Research Opportunities Programme in Science) during the summer break of my first year in University under Associate Prof. Yang Lei. On 10 July 2025, I attended a talk at S17 04-04 by Prof. Nimish Shah from The Ohio State University which was titled *Equidistribution of Expanding Degenerate Manifolds in the Space of Lattices*. I thought it was *cool* to attend, but Prof. Yang was in China for a conference back then. I got a glimpse of Lie Theory. More recently, I attended Prof. Yang's talk titled *Effective Equidistribution in Homogeneous Spaces and Restricted Projection Theorems* at the Institute of Mathematical Sciences (IMS) recently as part of the conference 'Arithmetic Dynamics and Diophantine Geometry', and I enjoyed it too.

Throughout, we will let $\mathcal{M}_{n\times n}(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries ($\mathbb{R}$ can be replaced with any field F such as the complex numbers $\mathbb{C}$). Define the orthogonal group

$$O(n,\mathbb{R}) = \left\{\mathbf{A} \in \mathcal{M}_{n\times n}(\mathbb{R}) : \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}\right\} \quad \text{which consists of orthogonal matrices.}$$

The elements of the orthogonal group $O(n)$ (a compact notation) are real linear isometries of $\mathbb{R}^n$. As a subset (in fact, a subgroup) of $O(n)$, we have the special orthogonal group $SO(n,\mathbb{R})$ which consists of orientation-preserving real linear isometries. That is,

$$SO(n) = \{\mathbf{A} \in O(n) : \det(\mathbf{A}) = 1\}.$$

As a generalisation of $O(n)$, we have the unitary group $U(n,\mathbb{C})$, which is defined as follows:

$$U(n) = \{\mathbf{U} \in \mathcal{M}_{n\times n}(\mathbb{C}) : \mathbf{U}^*\mathbf{U} = \mathbf{I}\} \tag{1}$$

The notation * in (1) means that we take the conjugate transpose of $\mathbf{U}$. That is, we first take the complex conjugate of every entry, then take the transpose of the matrix (in fact, either operation can go first). The elements are complex linear isometries of $\mathbb{C}^n$ for the standard Hermitian inner product. We then define the special unitary group as follows:

$$SU(n) = \{\mathbf{U} \in U(n) : \det(\mathbf{U}) = 1\}$$

All four are matrix Lie groups – they are closed under multiplication and inversion and naturally smooth *manifolds*. What do we mean by a manifold? A manifold is a space that locally looks like the Euclidean space $\mathbb{R}^n$. A classic example involves the Earth — it is globally curved but small patches look flat like the two-dimensional Euclidean space $\mathbb{R}^2$. Formally, we say that a space M is an n-manifold if it is locally Euclidean of dimension n, Hausdorff, and second countable (need to be familiar with these terms from MA3209 Metric and Topological Spaces first). To perform the usual (multivariable) calculus on manifolds, we need coordinate changes to be smooth. A smooth manifold is a manifold whose transition maps are smooth. At each point $p \in M$, you have a tangent space T_pM, an n-dimensional vector space capturing the behaviour near p. Once you have tangent spaces,

[1] In honour of the Norwegian Mathematician Sophus Lie (1842-1899).

you can define vector fields, perform integration, etc.

A Lie group is a group that is also a smooth manifold, with multiplication and inversion smooth. This brings tools of Calculus (tangent spaces, exponentials) to Group Theory. As mentioned, orthogonal and unitary groups are classic examples. We have the following results:

$$\dim\left(O(n)\right)=\dim\left(SO(n)\right)=\frac{n(n-1)}{2}\quad\text{and}\quad\dim\left(U(n)\right)=n^2\quad\text{and}\quad\dim\left(SU(n)\right)=n^2-1$$

These are not difficult to prove. For example, to show that the dimension of the orthogonal group $O\left(n\right)$ is $\frac{1}{2}n\left(n-1\right)$, fix some matrix $\mathbf{A}\in O\left(n\right)$. By definition, $\mathbf{A}^{\mathrm{T}}\mathbf{A}=\mathbf{I}$. Thereafter, construct a basis for $O\left(n\right)$, and it turns out that the upper triangular entries excluding the diagonal ones suffices.

A key property of matrix Lie groups that we will consider is compactness. We say that a matrix Lie group $G\subseteq\mathrm{GL}\left(n,\mathbb{C}\right)$ (recall we briefly mentioned in class that the general linear group $\mathrm{GL}\left(n,F\right)$ denotes the set of $n\times n$ invertible matrices with entries in a field F) is *compact* if it is compact in the usual topological sense as a subset of $\mathcal{M}_{n\times n}\left(\mathbb{C}\right)\cong\mathbb{R}^{2n^2}$. Here, $\cong$ means that the two sets are *isomorphic* — there exists some bijective transformation between the sets. In light of the Heine-Borel theorem from MA2108 Mathematical Analysis 1, a matrix Lie group G is compact if and only if it is closed as a subset of $\mathcal{M}_{n\times n}\left(\mathbb{C}\right)$ and bounded[2].

In general, the Lie algebra $\mathfrak{g}=\mathrm{Lie}\left(G\right)$ is the tangent space at the identity equipped with the commutator Lie bracket $[X,Y]=XY-YX$. For example, the Lie algebra of $SO\left(n\right)$ is

$$\mathfrak{so}\left(n\right)=\left\{\mathbf{X}\in\mathcal{M}_{n\times n}\left(\mathbb{R}\right):\mathbf{X}^{\mathrm{T}}+\mathbf{X}=\mathbf{0}\right\}$$

which is equivalently the set of real skew-symmetric matrices. As another example, the Lie algebra of $U\left(n\right)$ is

$$\mathfrak{u}\left(n\right)=\left\{\mathbf{X}\in\mathcal{M}_{n\times n}\left(\mathbb{C}\right):\mathbf{X}^{*}+\mathbf{X}=\mathbf{0}\right\}$$

which denotes the set of complex skew-Hermitian matrices. We can think of Hermitian matrices as a generalisation of symmetric matrices, and unitary, Hermitian, skew-Hermitian matrices as examples of normal matrices. Lastly,

$$\mathfrak{su}\left(n\right)=\left\{\mathbf{X}\in\mathfrak{u}\left(n\right):\mathrm{tr}\left(\mathbf{X}\right)=0\right\}$$

is the Lie algebra of $SU\left(n\right)$. A small concrete example is

$$\mathfrak{so}\left(2\right)=\left\{\begin{pmatrix}0&-a\\a&0\end{pmatrix}:a\in\mathbb{R}\right\}.$$

Intuitively, elements $\mathbf{X}$ of these Lie algebras are the infinitesimal generators of the one-parameter subgroups $t\mapsto\exp\left(tX\right)$ that stay within the group. Using these, one can deduce the mentioned formulae for the associated Lie algebras of the matrix groups. We now formally introduce the exponential map for Lie algebras. One should recall from Calculus that the usual exponential map

$$\exp:\mathbb{R}\to\mathbb{R}^{+}\text{ where }x\mapsto\sum_{n=0}^{\infty}\frac{x^n}{n!}\quad\text{converges for all }x\in\mathbb{R}.$$

Here, we used the Taylor series expansion of $\exp\left(x\right)$. As expected, we can extend this idea to matrices. Suppose $\mathbf{X}\in\mathcal{M}_{n\times n}\left(\mathbb{C}\right)$. Then, the exponential of $\mathbf{X}$ is given by the usual Taylor series expansion

$$\exp\left(\mathbf{X}\right)=\sum_{n=0}^{\infty}\frac{\mathbf{X}^n}{n!}$$

[2] Indeed, one can relate to sequences in $\mathbb{R}$.

which converges for all $\mathbf{X} \in \mathcal{M}_{n\times n}(\mathbb{C})$ and we can regard $\exp(\mathbf{X})$ as a *continuous function* in terms of $\mathbf{X}$. For example, if $\mathbf{X} \in \mathfrak{so}(n)$, then $\exp(\mathbf{X}) \in SO(n)$. The proof uses the fact that $\det(\exp(\mathbf{A})) = \exp(\operatorname{tr}(\mathbf{A}))$. For example, recall the Lie algebra $\mathfrak{so}(2)$. Exponentiation gives rotation by an angle at.

Compact lie groups, especially $SO(n)$, $SU(n)$, and $U(n)$, are *fundamental* in the general theory of Lie groups. We give two classical results in Algebraic Topology, which are

$$\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z} \text{ for } n \geq 3 \quad \text{and} \quad \pi_1(SU(n)) \cong \mathbb{Z} \text{ for } n \geq 2. \tag{2}$$

In general, for a topological space X with a chosen basepoint $x_0 \in X$, the group is written as $\pi_1(X, x_0)$ and it consists of homotopy classes of loops based at x_0. Intuitively, π_1 encodes the different ways one can loop around in X without being able to shrink the loop continuously to a point. For example, $\pi_1(S^1) \cong \mathbb{Z}$ since loops around a circle can wind around any integer number of times; $\pi_1(S^2)$ is trivial as every loop on a sphere can be shrunk to a point; $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$ since puncturing $\mathbb{R}^2$ leaves a hole. Suppose a topological space X is path-connected. That is to say, all choices of basepoint give isomorphic group. As such, we can simply write $\pi_1(X)$ without specifying the basepoint.

The proofs of (2) are out of my reach, though they can be found in p. 377 of the reference text used, which is *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction* by B. C. Hall. I briefly read that to prove $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$, one can prove some equivalent statement and eventually show that it suffices to prove the claim for $n = 3$. There is a nice geometric proof for this case and it is known as the *belt trick* (see Wikipedia for more information).