MA5211 Lie Theory

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These notes are based off **Dr. Emile Okada's** MA5211 Lie Theory materials.

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Chapter 1

From Lie Groups to Lie Algebras

1.1 Some Definitions

Definition 1.1 (Lie group). A Lie group is a group G which is also a finite-dimensional smooth manifold such that the group operations

$$G \times G \to G \text{ where } (x,y) \mapsto xy \quad \text{and} \quad G \to G \text{ where } x \mapsto x^{-1}$$

are smooth maps.

We give some remarks for Definition 1.1. Firstly, smooth means infinitely differentiable, i.e. $\in \mathcal{C}^\infty$. Also, an n-dimensional smooth manifold is locally $\mathbb{R}^n$ with $\mathcal{C}^\infty$ transition functions or change of local coordinates. In Definition 1.1, one may replace the smooth condition by a weaker one, say $\mathcal{C}^2$ or the stronger analytic condition. These turn out to be equivalent. This is related to Hilbert's fifth problem which asks whether locally Euclidean topological groups are Lie groups. The answer is affirmative.

Lastly, if the manifold is a complex manifold with group operations holomorphic, then we have a complex Lie group, which may be considered as a real Lie group of twice the dimension. In general when we talk about Lie groups, we mean real Lie groups.

Example 1.1 (real and complex vector space). Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, viewed as an additive group $(V,+)$. It is known that V is a finite-dimensional smooth manifold. Well, if V is a real vector space of dimension n, choose a basis and identify $V \cong \mathbb{R}^n$. This gives V the structure of an n-dimensional smooth manifold. On the other hand, if V is a complex vector space of complex dimension n, choose a complex basis and identify $V \cong \mathbb{C}^n$. As a real manifold, $\mathbb{C}^n \cong \mathbb{R}^{2n}$ so V is a $(2n)$-dimensional smooth manifold over $\mathbb{R}$.

Here, the group operations are addition

$$m : V \times V \to V \quad \text{where} \quad m(x,y) = x + y$$

and inversion inversion

$$i : V \to V \quad \text{where} \quad i(x) = -x.$$

Example 1.2 (general linear group). Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Let $\mathrm{GL}(V)$ denote the set of invertible linear transformations of V. From MA2202 Algebra I, we have seen that $\mathrm{GL}(V)$ is a group under composition. Let $\mathrm{End}(V)$ denote the set of endomorphisms of V. This denotes the vector space of all linear transformations of V. As an open subset of $\mathrm{End}(V)$, $\mathrm{GL}(V)$ is a smooth manifold.

Since multiplication can be regarded as a quadratic polynomial map (using matrix representation through a standard basis) and taking inverse is a smooth operation by Cramer's rule in MA2001 Linear Algebra I, by Definition 1.1, $\mathrm{GL}(V)$ is a Lie group. In fact, it is given the following special name — the general linear group. If $\dim(V) = n$, then $\mathrm{GL}(V)$ may be identified with the group of $n \times n$ invertible matrices, and will be denoted by either $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$.

Example 1.3 (Cartan's theorem). Cartan's theorem (Theorem **??**) which will be formally discussed in due course states that a closed subgroup of a Lie group is also a Lie group.

Based on our discussion in Example 1.3, we have some other interesting examples. Lie groups which are closed subgroups of $\mathrm{GL}(V)$ are called matrix groups (Example 1.4).

Example 1.4 (matrix groups). We give some examples of matrix groups.

(i) The special linear group $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det(A) = 1\}$

(ii) The orthogonal group $O(n) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : A^{\mathrm{T}}A = I_n\right\}$

(iii) The unitary group $U(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) : A^*A = I_n\}$[1]

(iv) The special orthogonal group $\mathrm{SO}(n) = \mathrm{SL}\{n,\mathbb{R}\} \cap O(n)$

(v) The special unitary group $\mathrm{SU}(n) = \mathrm{SL}(n,\mathbb{C}) \cap U(n)$

Let $D = \mathbb{R},\mathbb{C},\mathbb{H}$ be one of the three division algebras over $\mathbb{R}$ (here, $\mathbb{H}$ denotes the quaternions), with the following involution:

$$\sigma = \begin{cases} \text{identity} & \text{if } D = \mathbb{R}; \\ \text{complex conjugation} & \text{if } D = \mathbb{C}; \\ \text{quaternionic conjugation} & \text{if } D = \mathbb{H}. \end{cases} \tag{1.1}$$

For notational convenience, we write $\sigma(\lambda) = \overline{\lambda}$ for $\lambda \in D$.

Definition 1.2 (sesquelinear form). As per our discussion above, let $D = \mathbb{R},\mathbb{C},\mathbb{H}$ be one of the three division algebras over $\mathbb{R}$ with the involution as in (1.1). Let $V = D^n = D \oplus \cdots \oplus D$ denote the direct sum of n copies of D. A map $V \times V \to D$ is a sesquelinear form $\langle \cdot,\cdot \rangle$ on V if it is additive in each variable, and for every

[1] Recall that A^* denotes the conjugate transpose of A.

$v_1, v_2 \in V$ and $\lambda \in D$,

$$\langle \lambda v_1, v_2 \rangle = \lambda \langle v_1, v_2 \rangle \quad \text{and} \quad \langle v_1, \lambda v_2 \rangle = \overline{\lambda} \langle v_1, v_@ \rangle .$$

Definition 1.3 (types of sesquelinear forms). As per our discussion in Definition 1.2, a sesquelinear form $\langle \cdot, \cdot \rangle$ on V is

(i) non-degenerate if $\langle v_1, v \rangle = 0$ for all $v \in V$, then $v_1 = 0$;

(ii) Hermitian if $\langle v_2, v_1 \rangle = \overline{\langle v_1, v_2 \rangle}$;

(iii) skew-Hermitian if $\langle v_2, v_1 \rangle = -\overline{\langle v_1, v_2 \rangle}$

Definition 1.4 (isometry group). As per our discussion in (1.2) and (1.3), given a non-degenerate Hermitian or skew-Hermitian form $\langle \cdot, \cdot \rangle$ on V, the isometry group of $\langle \cdot, \cdot \rangle$ is defined to be

$$G = \{ g \in \mathrm{GL}(V, D) : \langle g v_1, g v_2 \rangle = \langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in V \} .$$

Example 1.5 (non-degenerate Hermitian form). Let $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be one of the three division algebras over $\mathbb{R}$ with the involution as in (1.1). Let V denote the direct sum of n copies of D. For non-negative integers p and q such that $p + q = n$, define the following non-degenerate Hermitian form on $V = D^n$ as follows:

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_p \overline{w}_p - z_{p+1} \overline{w}_{p+1} - \cdots - z_n \overline{w}_n ,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. By Definition 1.4, its isometry group is denoted by $O(p,q), U(p,q), \mathrm{Sp}(p,q)$ for $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and is respectively called the orthogonal, unitary, and symplectic groups of signature (p,q).

Example 1.6 (non-degenerate skew-symmetric bilinear form over $\mathbb{R}$). Let $D = \mathbb{R}$ be a division algebra over $\mathbb{R}$ with the involution as in (1.1). Let V denote the direct sum of n copies of D. A non-degenerate skew-symmetric bilinear form over $\mathbb{R}$ exists only on an even dimensional space. So, let $n = 2m$, and define the following form on $V = \mathbb{R}^{2m}$:

$$\langle x, y \rangle = x_1 y_{m+1} + \cdots + x_m y_{2m} - x_{m+1} y_1 - \cdots - x_{2m} y_m ,$$

where $x = (x_1, \ldots, x_{2m})$ and $y = (y_1, \ldots, y_{2m})$. Its isometry group is denoted by $\mathrm{Sp}(2m, \mathbb{R})$ and is called the real symplectic group of rank n. We may write the form

$$\langle x, y \rangle = x^{\mathrm{T}} J_m y \quad \text{where} \quad J_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} . \tag{1.2}$$

Then, its isometry group has the following matrix form:

$$\mathrm{Sp}(2m, \mathbb{R}) = \left\{ g \in \mathrm{GL}(2m, \mathbb{R}) : g^{\mathrm{T}} J_m g = J_m \right\} . \tag{1.3}$$

Example 1.7 (non-degenerate skew-Hermitian form). Let $D = \mathbb{H}$ be a division algebra over $\mathbb{R}$ with the involution as in (1.1). Let V denote the direct sum of n copies of D. Consider the following non-degenerate skew-Hermitian form on $V = \mathbb{H}^n$:

$$\langle z, w \rangle = z_1 j \overline{w}_1 + \cdots + z_n j \overline{w}_n,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and j is a pure quaternion, i.e. $\overline{j} = -j$. Its isometry group is denoted by $O^*(2n)$, known as the quaternionic orthogonal group.

Example 1.8 (symmetric and skew-symmetric forms). We have other variations, such as the isometry group of a non-degenerate symmetric form over $\mathbb{C}$, denoted by

$$O(n, \mathbb{C}) = \left\{ g \in \mathrm{GL}(n, \mathbb{C}) : g^{\mathrm{T}} g = I_m \right\},$$

and the isometry group of a non-degenerate skew-symmetric form over $\mathbb{C}$, denoted by

$$\mathrm{Sp}(2m, \mathbb{C}) = \left\{ g \in \mathrm{GL}(2m, \mathbb{C}) : g^{\mathrm{T}} J_m g = J_m \right\}, \tag{1.4}$$

where the matrix J_m was defined in (1.2).

So, we have the general linear group (Example 1.2), the isometry groups of symmetric or skew-symmetric (Examples 1.6 and 1.8) as well as Hermitian or skew-Hermitian forms (Examples 1.5 and 1.7) over $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Those groups, together with their *sister groups* with the names of special linear groups, special orthogonal, and special unitary groups (Example 1.4), are all termed classical groups. We remark that an element of $\mathrm{Sp}(2m, \mathbb{R})$ (1.3) or $\mathrm{Sp}(2m, \mathbb{C})$ (1.4 automatically has determinant 1, so there is no such thing as a special symplectic group.

1.2 A Toolkit on Differential Geometry

Definition 1.5 (tangent space). Let M be a differentiable manifold. Then, a differentiable curve α is a differentiable map

$$\alpha : (a, b) \to M \quad \text{where } a, b \in \mathbb{R}.$$

We may assume that $0 \in (a, b)$. For $m \in M$, the tangent space of M at m, denoted by $T_m M$, is the vector space of velocity vectors $\alpha'(0)$, where α is a differentiable curve passing through $m \in M$.

An intrinsic definition of a tangent vector at m may be given as a derivation on the space of $\mathcal{C}^\infty$ functions on M around m, namely a linear functional D such that

$$D(f_1 f_2) = D(f_1) f_2(m) + f_1(m) D(f_2) \quad \text{where } f_i \in \mathcal{C}^\infty(M). \tag{1.5}$$

Define an operator

$$D_\alpha : \mathcal{C}^\infty(M) \to \mathbb{R} \quad \text{where} \quad D_\alpha(f) = \left. \frac{d}{dt} \right|_{t=0} (f \circ \alpha)(t).$$

Then, D_α is a derivation since it is $\mathbb{R}$-linear and it satisfies (1.5), and in fact D_α depends only on the velocity vector $\alpha'(0)$ by the chain rule. Now, let $f : M \to N$ be a smooth map and $m \in M$. For a tangent vector $X \in T_m M$, choose any smooth curve α with $\alpha(0) = m$ and $\alpha'(0) = X$. Define a linear map

$$(df)_m = T_m M \to T_{f(m)} N$$

by requiring that

$$(df)_m(X) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \alpha)(t),$$

i.e. $(df)_m(X)$ is the velocity at $t = 0$ of the curve $f(\alpha(t))$ in N. This definition is independent of the choice of α with the same initial velocity X, and the resulting map $(df)_m$ is called the differential of f at m.

Definition 1.6 (differentiable vector field). Let M be a differentiable manifold. A differentiable vector field v on M is a differentiable section of the tangent bundle of M. That is, in a differentiable way,

$$M \ni m \mapsto v(m) \in T_m(m).$$

Definition 1.7 (integral curve). Let M be a differentiable manifold and v be a differentiable vector field on M. An integral curve of v is a curve

$$\alpha : (a,b) \to M \quad \text{such that} \quad \alpha(t) = v(\alpha(t)) \text{ for } t \in (a,b).$$

It is well-known that the first-order differential equation with initial condition $\alpha(t) = v(\alpha(t))$ where $\alpha(0) = m$ has a unique solution in a neighbourhood of 0, which we denote by $\Phi(t,m)$. Here, $t \in (a,b)$ for some $a, b \in \mathbb{R}$ and we may assume that (a,b) is the maximal interval on which the solution curve exists. Thus, we have a map

$$\Phi : (a,b) \times M \to M.$$

We have $\Phi(0,m) = m$ and

$$\Phi(t, \Phi(s,m)) = \Phi(t+s, m) \tag{1.6}$$

whenever either side is well-defined. This is because for a fixed s, each of them is the integral curve of v passing through $\Phi(s,m)$. If we write

$$\Phi_t : M \to M \quad \text{where} \quad m \mapsto \Phi(t,m),$$

then we have $\Phi_0 = \mathrm{id}$ and (1.6) may be rewritten as $\Phi_t \Phi_s = \Phi_{t+s}$, whenever either side is well-defined. The map Φ_t will be called the local flow associated with the vector field v.

Definition 1.8. Let G be a group and M be a set. We say that G acts on M or M is a

G-space if there exists a map

$$\Phi : G \times M \to M$$

such that $\Phi(e,x) = x$ and $\Phi(g,\Phi(h,x)) = \Phi(gh,x)$ for $x \in M$ and $g,h \in G$. Equivalently, let $\mathrm{Bi}(M)$ denote the group of bijective maps on M with composition as the group operation. Then, we have a group homomorphism $G \to \mathrm{Bi}(M)$, where $g \mapsto \Phi_g$.

1.3 The Exponential Map

1.4 Connected Lie Groups

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