MA4221 Partial Differential Equations

Thang Pang Ern

These notes are based off **Prof. Yao Yao** and **Dr. Wang Yamin's** MA4221 Partial Differential Equations materials.

This set of notes was last updated on May 15, 2026. If you would like to contribute a nice discussion to the notes or point out a typo, please send me an email at thangpangern@u.nus.edu.

Contents

Chapter 1

Origin of Partial Differential Equations

1.1 Introduction

Roughly speaking, the courses studies the following aspects of partial differential equations:

(i) Derivation of PDEs from physical principles

(ii) Techniques for solving some basic PDEs

(iii) Qualitative behaviour of solutions to PDEs

Typical examples include the following:

- Vibrations of strings or membranes, leading to the wave equation
- Heat transfer or diffusion, leading to the heat equation
- Stationary waves or stationary heat distributions, leading to Laplace's equation

Some important techniques include:

- the method of characteristics;
- separation of variables;
- Fourier methods, including Fourier series and Fourier transforms.

For qualitative behaviour, we will study ideas such as uniqueness of solutions, stability of solutions, the maximum principle, and finite speed of propagation for wave equations.

We first do a review of MA3220 Ordinary Differential Equations. An ordinary differential equation, or ODE, is a differential equation involving functions of only one independent variable and one or more of their derivatives with respect to that variable. In general, an ODE can be written in the form

$$F\left(x, u(x), u'(x), \ldots, u^{(k)}(x)\right) = 0,$$

where $u : \mathbb{R} \to \mathbb{R}$ is an unknown function of x, $u^{(k)}$ denotes the k^{th} derivative of u, and F is a given function. In contrast to ODEs, the key defining property of a PDE is that the unknown function depends on more than one independent variable. For example, if $u = u(x,y)$, then $u_x + u_y = x^2$ is a PDE, where we recall the notion of taking partial derivatives from MA2104 Multivariable Calculus.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be open, where $n \geq 1$. Suppose $u : \Omega \to \mathbb{R}$ is an unknown function, and let $\mathbf{x} \in \Omega$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write

$$\partial^\alpha u(\mathbf{x}) = \frac{\partial^{|\alpha|} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.$$

The length of α is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We denote by

$$\partial^i u(\mathbf{x}) = \{\partial^\alpha u(\mathbf{x}) : |\alpha| = i\}$$

the collection of all partial derivatives of order i.

An expression of the form

$$F\left(\partial^k u(\mathbf{x}), \ldots, \partial u(\mathbf{x}), u(\mathbf{x}), x\right) = 0 \quad \text{where } \mathbf{x} \in \Omega$$

is called a k^{th} order partial differential equation. Here F is a given function and u is the unknown function.

For example, if $u = u(x,y)$ and $k = 2$, then the second-order partial derivatives include u_{xx}, u_{xy}, u_{yy}. Just like ODEs, the order of a PDE is the highest order of derivatives appearing in the equation. For example, $u_{xx} + u_{yy} = 0$ is a second-order PDE and the equation $u_t + uu_y + u_{xxx} = 0$ is a third-order PDE.

If the unknown function u depends on two independent variables x and y, then the general form of a first-order PDE is

$$F(x, y, u, u_x, u_y) = 0.$$

On the other hand, the general form of a second-order PDE is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

Recall that, under sufficient differentiability assumptions, mixed partial derivatives commute: $u_{xy} = u_{yx}$. This is known as Clairaut's theorem.

Definition 1.2 (classical solution). For a given PDE, a function is called a classical solution if it is differentiable as many times as needed in the equation and satisfies the equation identically.

In this course, we mainly focus on classical solutions. Thus, for a k^{th} order PDE, the solution is assumed to be at least k-times continuously differentiable, i.e. $u \in \mathcal{C}^k(\Omega)$. It may be difficult to find analytical solutions to a PDE. However, it is usually straightforward to verify whether a given function is a solution by substituting it into the equation.

Example 1.1. Consider the PDE

$$u_t + 2u_x = 0.$$

Let $f \in \mathcal{C}^1(\mathbb{R})$ and define $u(x,t) = f(x-2t)$. One can easily verify using the chain rule that this is a solution to the PDE. Note that just like ODEs, sometimes a PDE comes with boundary conditions or initial conditions. In this case, the solution must satisfy these conditions in addition to satisfying the equation itself.

Many physical problems can be modelled by a PDE together with auxiliary conditions, such as initial conditions or boundary conditions.

> Definition 1.3 (well-posed PDE). A problem for a PDE is called well-posed if the following three conditions hold:
>
> **(i)** A solution exists
>
> **(ii)** The solution is unique
>
> **(iii)** The solution depends continuously on the data given in the problem, such as initial data or boundary conditions

Since we mainly deal with linear equations, existence can often be shown by directly constructing a solution. Later, we will learn techniques for proving uniqueness and stability.

1.2 Linear Operators

> Definition 1.4 (operator). An operator L takes a function u as input and maps it to another function, denoted by Lu. If the operator involves u and its derivatives, then it is called a differential operator.

Example 1.2. For example, if $u = u(x,y)$, then

$$Lu = \frac{\partial u}{\partial y} = u_y$$

defines a differential operator. Another example is

$$Lu = u_x + u_y.$$

> Definition 1.5 (linear operator). An operator L is called linear if for any functions u, v and any constant c, we have
>
> $$L(u+v) = Lu + Lv \quad \text{and} \quad L(cu) = cLu.$$

Example 1.3. For example, if $Lu = xyu_{xy} - u_{yy}$, then L is a linear differential operator. However, if $Lu = u'' + uu'$, then L is non-linear because of the product term uu'. Similarly, the equation

$$u_{tt} - u_{xx} + u^3 = 0$$

is non-linear because of the term u^3.

Definition 1.6 (linear PDE). A PDE is called linear if it can be written in the form

$$Lu = g,$$

where L is a linear operator and g is a given function of the independent variables. Otherwise, the PDE is called non-linear.

If $g = 0$, then the linear PDE $Lu = 0$ is called homogeneous; otherwise it is called inhomogeneous.

1.3 Examples of PDEs

We classify the following PDEs according to their order and whether they are linear homogeneous, linear inhomogeneous, or non-linear.

(i) Wave equation in two dimensions: This states that

$$u_{tt} = u_{xx} + u_{yy} \quad \text{or equivalently} \quad u_{tt} - u_{xx} - u_{yy} = 0.$$

This is a second-order linear homogeneous PDE.

(ii) Heat equation in one dimension: This states that

$$u_t = u_{xx} + f(x) \quad \text{or equivalently} \quad u_t - u_{xx} = f(x).$$

This is a second-order linear inhomogeneous PDE.

(iii) Laplace's equation in two dimensions: This states that

$$u_{xx} + u_{yy} = 0,$$

which is a second-order linear homogeneous PDE.

(iv) Schrödinger's equation: This states that

$$iu_t = u_{xx} \quad \text{or equivalently} \quad iu_t - u_{xx} = 0.$$

This is a second-order linear homogeneous PDE.

(v) Burger's equation: This states that

$$u_t + uu_x = 0,$$

which is a first-order nonlinear PDE because of the product term uu_x.

We now state some properties satisfied by linear equations (Propositions 1.1 and 1.2).

Proposition 1.1 (superposition principle). Let L be a linear differential operator. Suppose u_1 and u_2 are both solutions to the homogeneous linear PDE $Lu = 0$. Then for any constants c_1, c_2, the function

$$c_1 u_1 + c_2 u_2 \quad \text{is also a solution.}$$

Proof. Since $Lu_1 = 0$ and $Lu_2 = 0$, by linearity of L, we have

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2 = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Therefore, $c_1 u_1 + c_2 u_2$ is also a solution to $Lu = 0$. □

Proposition 1.2 (adding homogeneous solutions to particular solutions). Let L be a linear differential operator. Suppose u solves the inhomogeneous linear PDE $Lu = g$ and v solves the corresponding homogeneous equation $Lv = 0$. Then, $u + v$ is also a solution to the inhomogeneous equation $Lu = g$.

Proof. We have $L(u+v) = Lu + Lv = g + 0 = g$. □

1.4 Reducing PDEs to ODEs

Some linear PDEs can be solved directly using ODE techniques. The main idea is to treat some variables as parameters and integrate with respect to the variable in which differentiation occurs. Note that the general solution to a k^{th} order linear ODE contains k arbitrary constants. In contrast, the general solution to a PDE usually contains arbitrary functions.

Example 1.4. To solve the PDE $u_{xx} = 0$, it is easy to see that the general solution is

$$u(x,y) = xf(y) + g(y).$$

Example 1.5. To solve the PDE $u_x = xe^{3y}$, it is easy to see that the general solution is

$$u(x,y) = \frac{x^2}{2} e^{3y} + f(y).$$

1.5 First-Order Linear PDEs

We now study the transport equation

$$au_x + bu_y = 0, \tag{1.1}$$

where $a, b \neq 0$ are constants and $u = u(x,y)$.

We first discuss the solution to (1.1) using the geometric method. Recall that for a unit vector $\mathbf{v}$, the quantity $\mathbf{v}\cdot\nabla\mathbf{u}$ is the directional derivative of $\mathbf{u}$ in the $\mathbf{v}$-direction. Now, observe that

$$au_x + bu_y = (a,b)\cdot(u_x,u_y) = (a,b)\cdot\nabla\mathbf{u}.$$

Thus $au_x + bu_y = 0$ says that the directional derivative of u in the direction (a,b) is zero. Therefore u is constant along every straight line parallel to the vector (a,b). The straight lines parallel to (a,b) have equation

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}.$$

Equivalently, $bx - ay = C$, where C is constant along each such line. Since u is constant along each line $bx - ay = C$, the general solution is

$$u(x,y) = f(bx - ay) \tag{1.2}$$

where f is an arbitrary differentiable function.

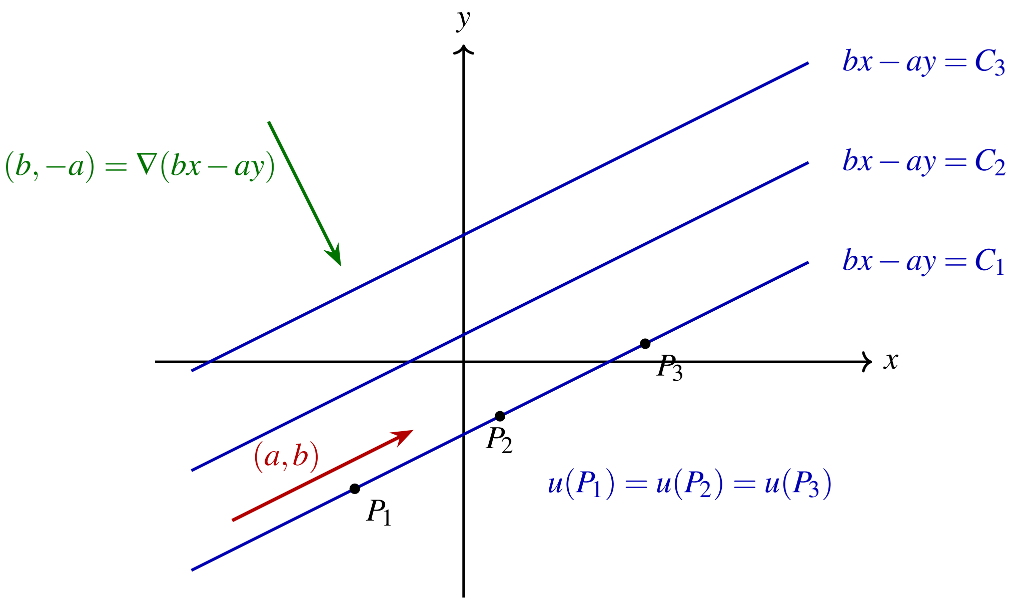


Figure 1.1: Characteristic lines for the transport equation $au_x + bu_y = 0$

When dealing with the method of characteristics, we used the fact that u is constant along every straight line parallel to $\mathbf{v} = (a,b)$. For the whole family of straight lines parallel to $\mathbf{v}$, u is constant on each line. The constant value may vary from one line to another. This family of curves is called the family of characteristics, or characteristic curves. The key idea is that if the value of u is known at one point along a characteristic curve, then the value of u is known along the entire curve.

We then provide an alternative method to solving (1.1) by changing coordinates. As expected, this method is known as the coordinate method. Choose new coordinates x', y' such that the x'-axis is parallel to (a,b), while the y'-axis is constant along the characteristic direction. Define

$$x' = ax + by \quad \text{and} \quad y' = bx - ay.$$

Using the chain rule, we have $u_x = au_{x'} + bu_{y'}$ and $u_y = bu_{x'} - au_{y'}$, so it is easy to see that (1.1) becomes

$$\left(a^2 + b^2\right) u_{x'} = 0.$$

Since $a^2 + b^2 \neq 0$, then $u_{x'} = 0$. So, u is independent of x', implying that $u = f(y')$. Substituting $y' = bx - ay$, we get (1.2).

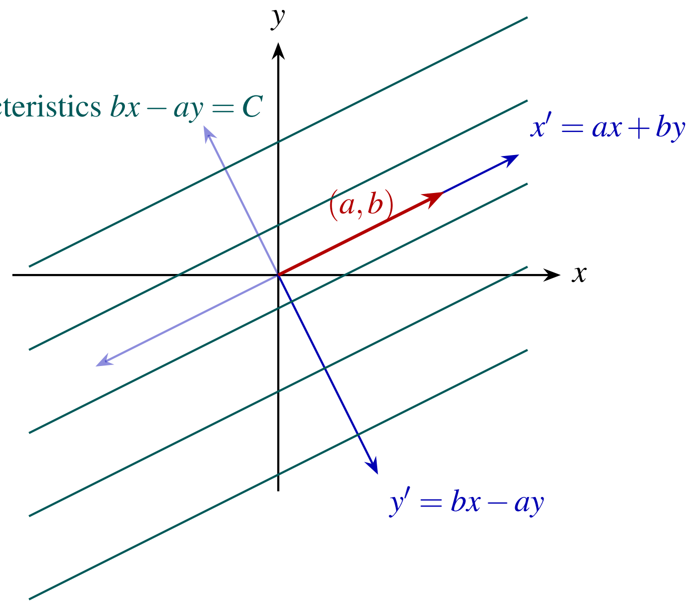


Figure 1.2: Change of coordinates for $au_x + bu_y = 0$

We now solve PDEs with auxiliary conditions.

Example 1.6. Find the solution to $3u_x + 2u_y = 0$ with auxiliary condition $u(x,0) = \cos x$.

Solution. Using (1.2), we have $u(x,y) = f(2x - 3y)$ being the general solution. Using the auxiliary condition, we have $f(2x) = \cos x$ so $f(s) = \cos\left(\frac{s}{2}\right)$. Hence, the solution is $u(x,y) = \cos\left(x - \frac{3}{2}y\right)$. □

Now, we discuss first-order linear PDEs with variable coefficients.

Example 1.7. Consider the PDE $u_x + yu_y = 0$ with auxiliary condition $u(1,y) = y^2$.

Solution. At each point (x,y), the PDE says that the directional derivative of u in the direction $(1,y)$ is 0. Thus, u is constant along integral curves of the vector field $(1,y)$. The characteristic curves satisfy

$$\frac{dy}{dx} = y$$

so $\frac{y}{e^x} = C$. Since u is constant along each characteristic, the general solution is

$$u(x,y) = f\left(\frac{y}{e^x}\right).$$

Now, use the auxiliary condition to obtain the solution, which is $u(x,y) = y^2 e^{2-2x}$. □

We now solve general first-order linear PDEs. That is to say, we consider equations of the form

$$a(x,y)\, u_x + b(x,y)\, u_y = f(u,x,y).$$

In this case, u is generally not constant along characteristic curves. However, the method of characteristics still works: along each characteristic curve, the PDE becomes an ODE.

Example 1.8. Solve the PDE $3u_x + 2u_y = -u$.

Solution. The characteristic curves satisfy $\frac{dy}{dx} = \frac{2}{3}$, so $y = \frac{2}{3}x + C$. Define

$$U(x) = u\left(x, \frac{2}{3}x + C\right).$$

By the chain rule, $U'(x) = \frac{1}{3}(3u_x + 2u_y)$. Then, using the PDE $3u_x + 2u_y = -u$, we obtain

$$U'(x) + \frac{1}{3}U(x) = 0.$$

Solving this ODE and using $C = y - \frac{2}{3}x$, we obtain

$$u(x,y) = f\left(y - \frac{2}{3}x\right)e^{-x/3}.$$

One can solve this using the change of coordinates method too, but we omit the solution. □

1.6 Derivation of the Transport Equation

Refer to Figure 1.3 throughout our discussion. Consider the transport of a certain substance, such as a pollutant in a river, in a one-dimensional channel. Let $u(x,t)$ denote the density of the substance at position x and time t. Thus $u(x,t)$ measures the amount of substance per unit length. Let $c(x,t)$ denote the velocity field at position x and time t. Assume that mass is neither created nor destroyed. The substance is only transported by the velocity field $c(x,t)$. What PDE governs the evolution of $u(x,t)$?

Let $I = [a,b]$ be an arbitrary interval. At time t, the total amount of substance in I is

$$m(t) = \int_a^b u(x,t)\,dx.$$

We compute $\frac{d}{dt}m(t)$ in two ways. On one hand, differentiating under the integral sign gives

$$\frac{d}{dt}m(t) = \int_a^b u_t(x,t)\,dx.$$

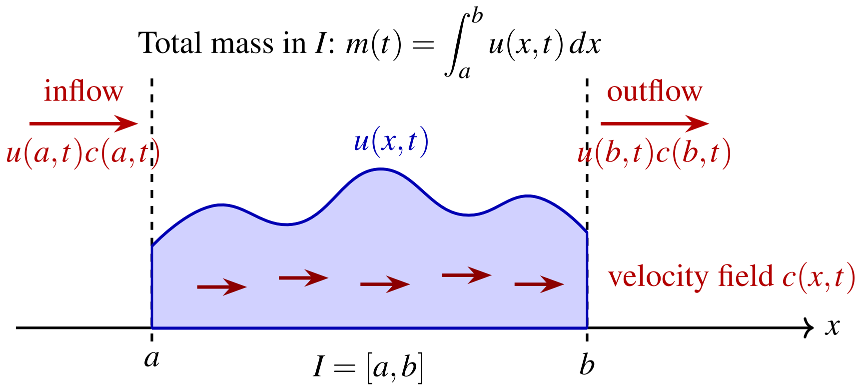


Figure 1.3: Derivation of the one-dimensional transport equation

On the other hand, the rate of change of mass in the interval is equal to the amount flowing in minus the amount flowing out. The flux at x is $u(x,t)c(x,t)$. Hence,

$$\frac{d}{dt}m(t) = u(a,t)c(a,t) - u(b,t)c(b,t).$$

Equivalently,

$$\frac{d}{dt}m(t) = -\left[u(x,t)c(x,t)\right]_{x=a}^{x=b}.$$

Using the fundamental theorem of calculus,

$$-\left[u(x,t)c(x,t)\right]_a^b = -\int_a^b \frac{\partial}{\partial x}\left(u(x,t)c(x,t)\right) dx.$$

Therefore

$$\int_a^b \left[u_t(x,t) + \frac{\partial}{\partial x}\left(c(x,t)u(x,t)\right)\right] dx = 0.$$

Since the interval $[a,b]$ is arbitrary, we obtain the PDE

$$u_t + (cu)_x = 0.$$

This is the one-dimensional transport equation in conservation form. If c is constant, then $(cu)_x = cu_x$ and the equation becomes

$$u_t + cu_x = 0.$$

Chapter 2

Distribution Theory

2.1 Introduction

Many problems in PDE require generalised notions of functions and derivatives. For instance, $u(x,t) = F(x+ct) + G(x-ct)$ solves the wave equation $u_{tt} = c^2 u_{xx}$, when F and G are sufficiently differentiable. But if F and G are not twice differentiable, the classical derivatives u_{tt} and u_{xx} may not exist. Distributions provide a framework in which such derivatives can still be defined.

Given a locally integrable function f on Ω, we associate to f a linear functional on test functions by

$$\varphi \mapsto \int_\Omega f(x)\varphi(x)\,dx.$$

This shifts the perspective: instead of studying f directly, we study how f acts on smooth compactly supported test functions.

The derivative can then be motivated by integration by parts. If f is differentiable and φ vanishes near $\partial\Omega$, then

$$\int_\Omega \frac{\partial f}{\partial x_j}(x)\varphi(x)\,dx = -\int_\Omega f(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx.$$

The right-hand side makes sense even if f is not differentiable. This motivates the definition of distributional derivatives.

Definition 2.1 (test function). Let $\Omega \subseteq \mathbb{R}^n$ be open. A function $\varphi : \Omega \to \mathbb{R}$ is called a test function if $\varphi \in C_c^\infty(\Omega)$. That is, φ is infinitely differentiable and has compact support in Ω. The support of φ is

$$\operatorname{supp}(\varphi) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}.$$

The collection of all test functions on Ω is denoted by $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$.

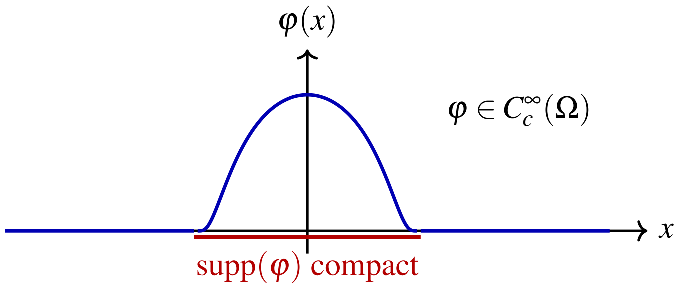


Figure 2.1: A test function is smooth and compactly supported

Definition 2.2 (distribution). A distribution, or generalised function, on Ω is a continuous linear map

$$f : \mathcal{D}(\Omega) \to \mathbb{R} \quad \text{where} \quad \varphi \mapsto (f, \varphi).$$

Linearity means

$$(f, a\varphi + b\psi) = a(f, \varphi) + b(f, \psi)$$

for all constants a, b and all test functions $\varphi, \psi \in \mathcal{D}(\Omega)$.

A distribution is thus a rule assigning a number to each test function.

Example 2.1 (locally integrable functions). Let f be locally integrable on $\mathbb{R}$. Define

$$(F, \varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx.$$

Prove that F is a distribution.

Solution. Linearity follows from the linearity of the integral. It is easy to see that

$$\begin{aligned}
(F, a\varphi + b\psi) &= \int_{-\infty}^{\infty} f(x)(a\varphi(x) + b\psi(x))\, dx \\
&= a\int_{-\infty}^{\infty} f(x)\varphi(x)\, dx + b\int_{-\infty}^{\infty} f(x)\psi(x)\, dx \\
&= a(F, \varphi) + b(F, \psi)
\end{aligned}$$

Continuity follows from local integrability and compact support. If the test functions vanish outside a common compact interval I and $\varphi_n \to \varphi$ uniformly, then

$$|(F, \varphi_n) - (F, \varphi)| = \left| \int_I f(x)(\varphi_n(x) - \varphi(x))\, dx \right| \le \max_{x \in I} |\varphi_n(x) - \varphi(x)| \int_I |f(x)|\, dx.$$

Since f is locally integrable, $\int_I |f(x)|\, dx < \infty$, and since $\varphi_n \to \varphi$ uniformly, the right-hand side tends to 0. □

We now discuss the convergence and derivative of distributions.

Definition 2.3 (weak convergence of distributions). Let f_n be a sequence of distributions and let f be another distribution. We say that f_n converges weakly to f if, for every test function φ,

$$(f_n, \varphi) \to (f, \varphi) \quad \text{as } n \to \infty.$$

Definition 2.4 (derivative of a distribution). For any distribution f, its derivative f' is defined by $(f', \varphi) = -(f, \varphi')$ for all test functions φ.

The derivative of a distribution always exists and is again a distribution. This is one of the main advantages of distribution theory: generalised functions are always differentiable in the distributional sense.

2.2 The Dirac Delta and Heaviside Function

The Dirac delta is not a classical function. Informally, one writes

$$\delta(x - x_0) \simeq \begin{cases} +\infty & \text{if } x = x_0; \\ 0 & \text{if } x \neq x_0 \end{cases}$$

with the normalisation

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1.$$

The precise definition is as a distribution.

Definition 2.5 (Dirac delta distribution). The delta function δ is the distribution defined by $(\delta, \varphi) = \varphi(0)$ for every test function φ. More generally, $(\delta_{x_0}, \varphi) = \varphi(x_0)$. It is common to write formally

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\, dx = \varphi(0) \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x - x_0)\varphi(x)\, dx = \varphi(x_0).$$

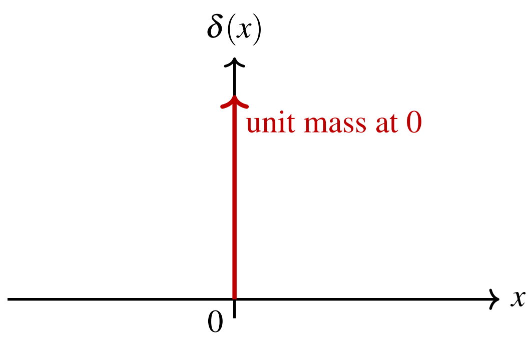


Figure 2.2: The Dirac delta is represented as a unit mass concentrated at a point.

Let H be the Heaviside function

$$H(x) = \chi_{(0,\infty)}(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$

Then, in the distributional sense, $H' = \delta$. Indeed, for every test function φ,

$$(H', \varphi) = -(H, \varphi') = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) = (\delta, \varphi).$$

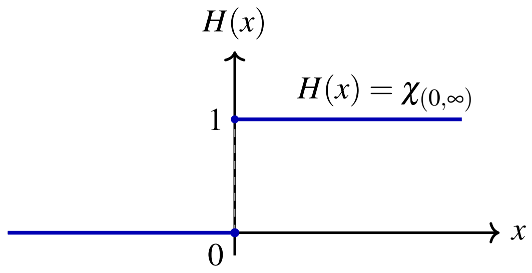


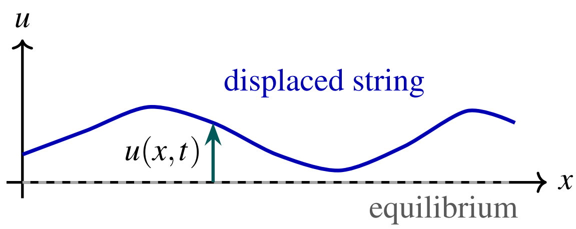
Figure 2.3: The derivative of the jump in the Heaviside function is the Dirac delta distribution

Chapter 3

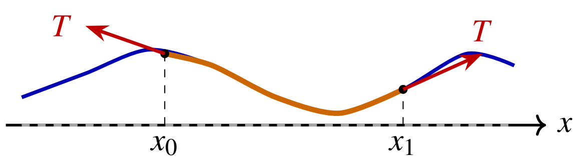
Waves and Diffusions

3.1 Derivation of the One-Dimensional Wave Equation

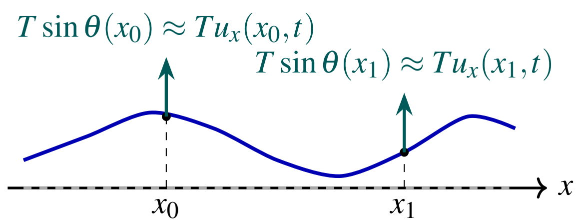
Consider the motion of a vibrating flexible elastic string with constant density ρ. Assume that the string remains in the xy-plane and that its equilibrium position lies along the x-axis. Let $u(x,t)$ denote the vertical displacement of the string from its equilibrium position. We neglect horizontal displacement and gravity. Here, 'one-dimensional' means that the spatial variable is one-dimensional. What PDE governs the evolution of $u(x,t)$?



Let T be the constant tension force along the string. Consider a small segment of the string between $x = x_0$ and $x = x_1$.



The vertical component of the tension force at position x is approximately $T\sin\theta$.



Net vertical force: $\text{Tu}_x(x_1,t) - Tu_x(x_0,t)$

Since $\tan\theta = u_x$, then

$$\sin\theta = \frac{u_x}{\sqrt{1+u_x^2}}.$$

Assuming that the displacement is small, so that $|u_x| \ll 1$, we have the approximation

$$\frac{u_x}{\sqrt{1+u_x^2}} \approx u_x.$$

Therefore the vertical component of the tension is approximately Tu_x. As such, the net vertical force acting on the segment $[x_0, x_1]$ is

$$Tu_x(x_1,t) - Tu_x(x_0,t) = Tu_x \,|_{x_0}^{x_1}.$$

Using the fundamental theorem of calculus,

$$T\, u_x|_{x_0}^{x_1} = T\int_{x_0}^{x_1} u_{xx}(x,t)\,dx.$$

On the other hand, by Newton's second law $F = ma$, the vertical force is also equal to

$$\int_{x_0}^{x_1} \rho u_{tt}(x,t)\,dx.$$

Thus

$$T\int_{x_0}^{x_1} u_{xx}(x,t)\,dx = \int_{x_0}^{x_1} \rho u_{tt}(x,t)\,dx.$$

Hence

$$\int_{x_0}^{x_1} (\rho u_{tt}(x,t) - Tu_{xx}(x,t))\,dx = 0.$$

Since the interval $[x_0, x_1]$ is arbitrary, we get $\rho u_{tt} - Tu_{xx} = 0$. Therefore

$$u_{tt} = \frac{T}{\rho} u_{xx}. \tag{3.1}$$

Setting $c^2 = \frac{T}{\rho}$, we obtain the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}. \tag{3.2}$$

Here c is called the wave speed.

3.2 Boundary Conditions

For the one-dimensional wave equation (3.1), we need to specify the domain of u. If the string extends infinitely in both directions, then $x \in \mathbb{R}$. If the string has finite length L, then $x \in [0, L]$. In this case, we need to specify what happens at the endpoints of the spatial domain. Such conditions are called boundary conditions.

If the string is fixed at both ends, then $u(0,t) = 0$ and $u(L,t) = 0$. More generally, if the endpoint displacement is prescribed, we may have $u(0,t) = f(t)$, where f is a given function of time.

We now discuss the three types of boundary conditions for PDEs. Let $D \subseteq \mathbb{R}^n$ be the spatial domain. Its boundary is denoted by ∂D.

(i) Dirichlet boundary condition: The value of u is specified on the boundary. That is, $u = g$ on ∂D.

(ii) Neumann boundary condition: The normal derivative of u is specified on the boundary. That is,

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial D.$$

Here, $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal vector on ∂D.

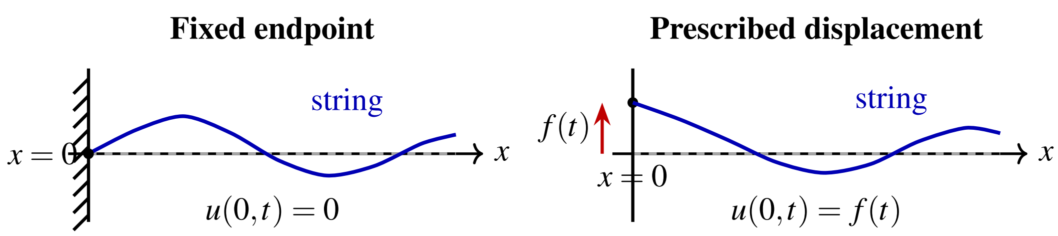
(iii) Robin boundary condition: A linear combination of u and its normal derivative is specified. That is,

$$\frac{\partial u}{\partial n} + \alpha u = h \quad \text{on } \partial D.$$

One can also impose mixed boundary conditions, such as Dirichlet boundary conditions on one part of the boundary and Neumann boundary conditions on another part.

We now discuss different boundary conditions for a vibrating string. Consider a vibrating string on the interval $[0,L]$. At an endpoint, for example $x = 0$, different physical assumptions lead to different boundary conditions. We consider three cases.

(i) Fixed endpoint or given displacement: If the endpoint is fixed, then $u(0,t) = 0$; if the endpoint has a prescribed displacement, then $u(0,t) = f(t)$, where f is a given function.



(ii) Endpoint freely sliding without friction: If the endpoint can freely slide up or down without friction, then the slope at the endpoint is zero. That is, $u_x(0,t) = 0$. This is a Neumann boundary condition.

(iii) Endpoint attached to a rubber band: If the endpoint can slide up or down and is tied to the equilibrium position by a rubber band, then the boundary condition has the form

$$ku(0,t) + Tu_x(0,t) = 0.$$

This is a Robin boundary condition.

Here is an extension to free boundary problems. A free boundary problem is a boundary value problem involving a PDE on a domain whose boundary is not fixed in advance. For example, consider a block of melting ice. The ice-water interface is unknown and evolves over time. This interface is called a free boundary. The ice shrinks depending on the temperature distribution. In such a problem, if $u(x,t)$ denotes temperature, one may have a condition such as $u > 0$ in the water region and $u = 0$ on the ice-water surface, the set

$$\partial \{(x,t) : u(x,t) > 0\}$$

is part of the unknown and represents the free boundary.

We now discuss boundary problems for the wave equation. One would need some knowledge of initial conditions prior to this (see Chapter 3.3). Consider the n-dimensional wave equation on a bounded domain $\Omega \subseteq \mathbb{R}^n$. Let $u = u(x,t)$ and $x = (x_1, \ldots, x_n) \in \Omega$. The wave equation is

$$\rho u_{tt} - T\Delta u = 0 \quad \text{where } x \in \Omega \text{ and } t > 0.$$

If the boundary is fixed, we impose the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{where } x \in \partial\Omega \text{ and } t > 0.$$

Since the wave equation is second order in time, we need two initial conditions, namely

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \quad \text{where } x \in \Omega.$$

Thus, a typical boundary-initial value problem for the wave equation is

$$\begin{aligned} \rho u_{tt} - T\Delta u &= 0 \text{if } x \in \Omega \text{ and } t > 0; \\ u(x,t) &= 0 \text{if } x \in \partial\Omega \text{ and } t > 0; \\ u(x,0) &= f(x) \text{if } x \in \Omega; \\ u_t(x,0) &= g(x) \text{if } x \in \Omega. \end{aligned}$$

3.3 Initial Conditions

For PDEs involving time, such as the wave equation, if the physical state is specified at an initial time t_0, the condition is called an initial condition. In general, if the equation is k^{th} order in the time variable, then a full set of initial conditions consists of the value of u and its time derivatives up to order $k-1$.

Example 3.1 (one-dimensional wave equation). For the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ as mentioned in (3.2), which is second order in time, one initial condition is needed: $u(x,t_0) = u_0(x)$.

Example 3.2 (one-dimensional transport equation). Recall the one-dimensional transport equation $u_t + (cu)_x = 0$ as mentioned in (1.1) which is first order in time. One initial condition is needed, which is $u(x,t_0) = u_0(x)$.

Example 3.3 (one-dimensional heat equation). For the heat equation $u_t = ku_{xx}$ which is first order in time, one initial condition is needed, which is $u(x,t_0) = u_0(x)$.

3.4 Green's Formulae and the Divergence Theorem

We recall the divergence theorem as in MA2104 Multivariable Calculus.

Theorem 3.1 (divergence theorem). Let $D \subseteq \mathbb{R}^2$ be a bounded spatial domain with piecewise C^1 boundary ∂D. Let $\mathbf{n}$ be the outward unit normal vector on ∂D. Let $\mathbf{F}$ be a C^1 vector field on $\overline{D} = D \cup \partial D$. Then

$$\iint_D \nabla \cdot \mathbf{F}(x,y)\, dxdy = \int_{\partial D} \mathbf{F}(x,y) \cdot \mathbf{n}\, dS.$$

Here $\nabla \cdot \mathbf{F}$ is the divergence of $\mathbf{F}$ and dS is the element of arc length on ∂D.

Theorem 3.2 (Green's formula). Let $D \subseteq \mathbb{R}^n$ be a bounded open subset with $\mathcal{C}^1$ boundary ∂D. Suppose $u = u(x_1, \ldots, x_n) \in \mathcal{C}^1\left(\overline{D}\right)$. Taking the vector field

$$\mathbf{F} = (0, \ldots, 0, u, 0, \ldots, 0),$$

where u is in the i^{th} component, the divergence theorem (Theorem 3.1) gives

$$\int_D u_{x_i}\, dx = \int_{\partial D} u n_i\, dS.$$

Here, $\mathbf{n} = (n_1, \ldots, n_n)$ is the outward unit normal vector on ∂D.

We now state the multi-dimensional integration by parts formula (Theorem 3.3).

Theorem 3.3 (multi-dimensional integration by parts). Let $u, v \in \mathcal{C}^1\left(\overline{D}\right)$. Then,

$$\int_D u_{x_i} v\, dx = -\int_D u v_{x_i}\, dx + \int_{\partial D} u v n_i\, dS.$$

Theorem 3.4 (Green's first identity). For sufficiently smooth functions u, v on a bounded domain $D \subseteq \mathbb{R}^n$, we have

$$\int_D \nabla u \cdot \nabla v\, dx = -\int_D v \Delta u\, dx + \int_{\partial D} v \frac{\partial u}{\partial n}\, dS.$$

Equivalently,

$$\int_D v \Delta u\, dx = -\int_D \nabla u \cdot \nabla v\, dx + \int_{\partial D} v \frac{\partial u}{\partial n}\, dS.$$

Theorem 3.5 (Green's second identity). For sufficiently smooth functions u, v on a bounded domain $D \subseteq \mathbb{R}^n$, we have

$$\int_D u \Delta v - v \Delta u\, dx = \int_{\partial D} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\, dS.$$

3.5 Derivation of the Two-Dimensional Wave Equation

We now derive the two-dimensional wave equation using a vibrating drumhead. Consider the motion of a vibrating flexible elastic drumhead with constant density ρ, whose equilibrium position lies in the xy-plane. Let $u(x,y,t)$ denote the vertical displacement of the drumhead. We assume there is no horizontal motion. What PDE governs the evolution of $u(x,y,t)$?

We reason similarly to the one-dimensional case as in Chapter 3.1. Let $D \subseteq \mathbb{R}^2$ be an arbitrary domain in the xy-plane. The vertical acceleration is u_{tt}. Therefore, by Newton's second law, the total vertical force on D is

$$F = \iint_D \rho u_{tt}(x,y,t)\, dxdy. \tag{3.3}$$

On the other hand, the vertical component of the tension force along the boundary is approximately $T\nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal to ∂D. Integrating along the whole boundary gives

$$F = \int_{\partial D} T\nabla u \cdot \mathbf{n}\, dS. \tag{3.4}$$

Using the divergence theorem (Theorem 3.1),

$$\int_{\partial D} T\nabla u \cdot \mathbf{n}\, dS = \iint_D \nabla \cdot (T\nabla u)\, dxdy.$$

Assuming T is constant, we have

$$\nabla \cdot (T\nabla u) = T\Delta u.$$

Hence,

$$F = \iint_D T\Delta u\, dxdy.$$

Combining this with (3.3), we get

$$\iint_D \rho u_{tt}\, dxdy = \iint_D T\Delta u\, dxdy.$$

Therefore,

$$\iint_D (\rho u_{tt} - T\Delta u)\, dxdy = 0.$$

Since D is arbitrary, we obtain $\rho u_{tt} - T\Delta u = 0$. Thus, $u_{tt} = \frac{T}{\rho}\Delta u$. Let $c^2 = \frac{T}{\rho}$. So, we obtain the two-dimensional wave equation

$$u_{tt} = c^2 (u_{xx} + u_{yy}).$$

We now state the boundary-initial value problem for the two-dimensional wave equation. Consider the two-dimensional wave equation on a bounded spatial domain $D \subseteq \mathbb{R}^2$. That is,

$$u_{tt} = c^2 \Delta u \quad \text{where } (x,y) \in D \text{ and } t > 0.$$

If the boundary of the drum is fixed, then we impose the Dirichlet boundary condition

$$u(x,y,t) = 0 \quad \text{where } (x,y) \in \partial D \text{ and } t > 0.$$

Since the wave equation is second order in time, we also need two initial conditions, say

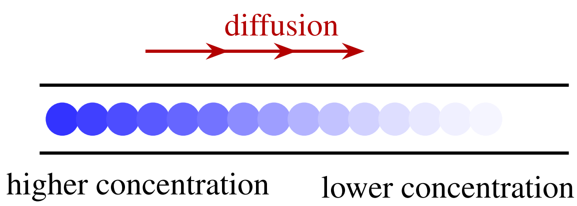
$$u(x,y,0) = \varphi(x,y) \text{ and } u_t(x,y,0) = \psi(x,y) \quad \text{where } (x,y) \in D.$$

Thus, the full boundary-initial value problem is

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{if } (x,y) \in D \text{ and } t > 0; \\ u = 0 & \text{if } (x,y) \in \partial D \text{ and } t > 0; \\ u(x,y,0) = \varphi(x,y) & \text{if } (x,y) \in D; \\ u_t(x,y,0) = \psi(x,y) & \text{if } (x,y) \in D. \end{cases}$$

3.6 Derivation of the One-Dimensional Heat Equation

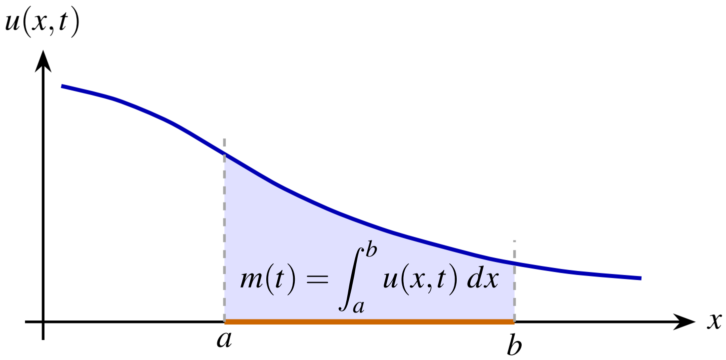
We now derive the one-dimensional heat equation. Consider diffusion along a one-dimensional pipe. Let $u(x,t)$ denote the concentration of some substance, such as dye in a liquid, at position x and time t. Here, concentration means the amount per unit length. The dye moves from regions of higher concentration to regions of lower concentration. Assume the liquid itself is motionless.



Fick's law states that the rate of diffusion across unit area is proportional to the negative concentration gradient. In one dimension, the flux is $-ku_x$, where $k > 0$ is the diffusion coefficient. What PDE governs the evolution of $u(x,t)$?

Let $[a,b]$ be an arbitrary interval. At time t, the total amount of substance in $[a,b]$ is

$$m(t) = \int_a^b u(x,t)\, dx.$$

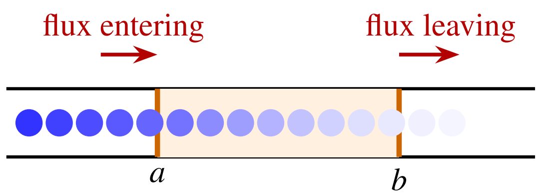


On one hand,

$$m'(t) = \int_a^b u_t(x,t)\,dx.$$

On the other hand, using Fick's law, the rate of change of mass in $[a,b]$ is equal to the flux entering at a minus the flux leaving at b. That is to say,

$$m'(t) = ku_x(b,t) - ku_x(a,t).$$



Using the fundamental theorem of calculus, we have

$$u_x(b,t) - u_x(a,t) = \int_a^b u_{xx}(x,t)\ dx$$

so

$$\int_a^b u_t - ku_{xx}\,dx = 0.$$

Since the interval $[a,b]$ was arbitrary, we obtain $u_t - ku_{xx} = 0$. Equivalently,

$$u_t = ku_{xx}$$

which is the one-dimensional heat equation.

3.7 Derivation of the Three-Dimensional Heat Equation

Now, consider diffusion in three-dimensional space. Let $u(x,y,z,t)$ denote the concentration of a substance, measured as amount per unit volume. Let $D \subseteq \mathbb{R}^3$ be an arbitrary spatial domain. The total amount of substance in D is

$$m(t) = \iiint_D u(x,y,z,t)\ dxdydz.$$

On one hand,

$$m'(t) = \iiint_D u_t(x,y,z,t)\ dxdydz. \tag{3.5}$$

On the other hand, by Fick's law, the flux vector is $-k\nabla u$. The rate of change of mass inside D is equal to the negative outward flux. That is,

$$m'(t) = \int_{\partial D} k\nabla u \cdot \mathbf{n}\, dS.$$

By the divergence theorem (Theorem 3.1),

$$\int_{\partial D} k\nabla u \cdot \mathbf{n}\, dS = \iiint_D k\nabla \cdot \nabla u\, dxdydz.$$

Since $\nabla \cdot \nabla u = \Delta u$, then

$$m'(t) = \iiint_D k\nabla u\, dxdydz. \tag{3.6}$$

Equating the two expressions for $m'(t)$, namely (3.5) and (3.6), we obtain

$$\iiint_D u_t\, dxdydz = \iiint_D k\nabla u\, dxdydz.$$

Hence,

$$\iiint_D u_t - k\Delta u\, dxdydz = 0.$$

Since D is arbitrary, we obtain the three-dimensional heat equation

$$u_t = k\Delta u.$$

3.8 Derivation of the Heat Equation from Heat Transfer

The heat equation can also be derived from heat transfer. Let $u(x,y,z,t)$ denote the temperature at the point (x,y,z) at time t. Let $H(t)$ denote the total heat, measured in calories, contained in a region D. Assume that no heat is generated inside D. If c denotes the heat capacity and ρ denotes the density, then

$$H(t) = \iiint_D c\rho u(x,y,z,t)\, dxdydz.$$

As such,

$$\frac{dH}{dt} = \iiint_D c\rho u_t(x,y,z,t)\, dxdydz. \tag{3.7}$$

Fourier's law for heat transfer says that heat flow across a surface is proportional to the negative temperature gradient and the surface area. Thus the heat flux vector is $-\kappa\nabla u$, where $\kappa > 0$ is the heat conductivity.

The change of heat energy in D is equal to the negative outward heat flux. That is,

$$\frac{dH}{dt} = \int_{\partial D} \kappa\nabla u \cdot \mathbf{n}\, dS.$$

By the divergence theorem (Theorem 3.1), we have

$$\int_{\partial D} \kappa \nabla u \cdot \mathbf{n}\, dS = \iiint_D \kappa \Delta u \, dxdydz.$$

So,

$$\frac{dH}{dt} = \iiint_D \kappa \Delta u \, dxdydz. \tag{3.8}$$

By equating (3.7) and (3.8), we have

$$\iiint_D c\rho u_t - \kappa \Delta u \, dxdydz = 0.$$

Since the region D was arbitrary, then

$$c\rho u_t = \kappa \Delta u \quad \text{or equivalently} \quad u_t = \frac{\kappa}{c\rho}\Delta u.$$

In the above derivation, we assumed that the heat conductivity κ is constant. If $\kappa = \kappa(x,y,z)$ is a function of position, then the heat equation becomes

$$c\rho u_t = \nabla \cdot (\kappa(x,y,z)\nabla u).$$

Now, let D be a given domain for the heat equation. We consider three cases.

(i) Prescribed boundary temperature: If the boundary temperature is prescribed, then

$$u = f(x,y,z,t) \quad \text{on } \partial D.$$

This is a Dirichlet boundary condition.

(ii) Insulated boundary: If the boundary is insulated, then no heat passes through the boundary. Hence, the normal derivative vanishes. That is,

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$$

This is a Neumann boundary condition.

(iii) Heat source: If there is a heat source f inside the domain, then the heat equation becomes

$$c\rho u_t = \kappa \Delta u + f(x,y,z,t).$$

This is a Robin boundary condition.

We now state boundary problems for the heat equation. Consider the n-dimensional heat equation on a bounded open set $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{C}^1$ boundary. Let $u : \Omega \times [0,\infty) \to \mathbb{R}$ solve the PDE

$$u_t - \Delta u = 0 \quad \text{where } x \in \Omega \text{ and } t > 0.$$

An initial condition is

$$u(x,0) = f(x) \quad \text{where } x \in \Omega.$$

We impose either the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{where } x \in \partial\Omega \text{ and } t > 0$$

or the Neumann boundary condition

$$u_\nu(x,t) = 0 \quad \text{where } x \in \partial\Omega \text{ and } t > 0.$$

Here, $u_\nu = \frac{\partial u}{\partial \nu}$ and ν is the outward unit normal vector on $\partial\Omega$. Thus the problem is

$$\begin{cases} u_t - \Delta u = 0 & \text{if } x \in \Omega \text{ and } t > 0; \\ u(x,0) = f(x) & \text{if } x \in \Omega; \\ u = 0 \text{ or } u_\nu = 0 & \text{if } x \in \partial\Omega \text{ and } t > 0. \end{cases}$$

We now give an application of Green's formulas to the heat equation. Suppose $u : \Omega \times [0,\infty) \to \mathbb{R}$ solves the heat equation $u_t - \Delta u = 0$ with either homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$, or homogeneous Neumann boundary condition $u_\nu = 0$ on $\partial\Omega$. Define

$$E(t) = \int_\Omega u^2 \, dx. \tag{3.9}$$

We claim that E is non-increasing in time.

Proof. Differentiating (3.9) under the integral sign yields

$$E'(t) = 2\int_\Omega u(x,t)\, u_t(x,t)\ dx.$$

Using the heat equation $u_t = \Delta u$, we obtain

$$E'(t) = 2\int_\Omega u\Delta u \, dx.$$

By Green's first identity (Theorem 3.4), we have

$$\int_\Omega u\Delta u \, dx = -\int_\Omega |\nabla u|^2 \ dx + \int_{\partial\Omega} u\frac{\partial u}{\partial \nu} \, dS.$$

Thus,

$$E'(t) = -2\int_\Omega |\nabla u|^2 \ dx + 2\int_{\partial\Omega} u u_\nu \, dS.$$

The boundary term vanishes in either case. That is to say, if $u = 0$ on $\partial\Omega$, then $uu_\nu = 0$; if $u_\nu = 0$ on $\partial\Omega$, then $uu_\nu = 0$. As such,

$$E'(t) = -2\int_\Omega |\nabla u|^2 \ dx \leq 0$$

which implies E is non-increasing in time. □

3.9 Laplace Equation

Recall from MA2104 Multivariable Calculus that

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}.$$

The wave equation is $u_{tt} = c^2 \Delta u$ and the heat equation is $u_t = k\Delta u$. For a PDE involving time t, if a solution is independent of time, then it is called a stationary solution or a steady state. To find steady states, we set all terms involving time derivatives equal to zero.

For the wave equation, a stationary solution satisfies $u_{tt} = 0$ so $\Delta u = 0$; for the heat equation, a stationary solution satisfies $u_t = 0$ so $\Delta u = 0$. Thus, steady states of the wave and heat equations lead to the Laplace equation, which states that $\Delta u = 0$. Solutions of the Laplace equation are called harmonic functions.

In one spatial dimension, we have $\Delta u = u_{xx}$ so the Laplace equation becomes $u_{xx} = 0$. Integrating twice yields $u(x) = A + Bx$. Hence in one dimension, the only harmonic functions are affine functions. In higher dimensions, harmonic functions are much richer and more interesting. We will explore them in Chapter **??**.

The inhomogeneous version of the Laplace equation is $\Delta u = f$, where f is a given function. This is called the Poisson equation.

3.10 Well-Posedness of PDEs

Definition 3.1 (well-posed problem). A PDE problem is said to be well-posed if it satisfies all of the following three conditions:

- **(i) Existence:** There exists at least one solution u satisfying the PDE and all the auxiliary conditions.
- **(ii) Uniqueness:** There is at most one solution.
- **(iii) Stability:** The unique solution u has continuous dependence on the auxiliary condition.

Example 3.4 (non-existence). Consider the transport equation

$$u_t + 5u_x = 0 \quad \text{where } x \in [0,1] \text{ and } t > 0,$$

with the overdetermined auxiliary conditions $u(x,0) = x$, $u(0,t) = 0$, and $u(1,t) = 1$. The characteristics meet conflicting auxiliary conditions, so there is no solution.

To see why, suppose $u(x,t) = f(x-5t)$ is a solution to the transport equation. Since $u(x,0) = x$, then $f(x) = x$ for $0 \le x \le 1$. Next, by considering $u(0,t) = 0$, we have

$f(-5t) = 0$ which forces $f(s) = 0$ for $s < 0$. We see that $u\left(1, \frac{1}{10}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$ but this contradicts the right boundary condition $u(1,t) = 1$.

Example 3.5 (non-uniqueness). Let $\Omega = \mathbb{R}^2 \setminus B(0,1)$, where $B(x,r)$ denotes the open ball of radius r centred at x. We illustrate that solutions to the following problem are non-unique:

$$\begin{cases} \Delta u = 0 & \text{for } x \in \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Looking for radial solutions $u(x) = v(r)$ where $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$, we can find the fundamental solution to the Laplace equation. The fundamental solution $\Phi(x)$ is defined as $-\frac{1}{2\pi}\log|x|$ for $n = 2$. For our domain, $u = c\log|x|$ satisfies the equation and boundary conditions because on the boundary $r = 1$, $\log 1 = 0$. The constant c can be chosen differently, showing the solution is not unique.

Example 3.6 (no stability). Consider the backward (in time) heat equation

$$u_t = -u_{xx} \quad \text{where } x \in [0,\pi] \text{ and } t > 0$$

with $u(x,0) = 0$ and $u(0,t) = u(\pi,t) = 0$. The trivial solution is $u = 0$. For any $n \in \mathbb{N}$, the function

$$u_n(x,t) = \frac{1}{n}e^{n^2 t}\sin(nx)$$

solves the heat equation with initial condition $u_n(x,0) = \frac{1}{n}\sin(nx)$. For large n, the maximum difference in initial data is bounded by $\frac{1}{n}$, which is small. However, at a short time $t = \frac{1}{n}$, the difference grows rapidly. That is,

$$\max\left|u_n\left(x, \frac{1}{n}\right) - u\left(x, \frac{1}{n}\right)\right| = \frac{1}{n}e^n \gg 1.$$

Thus, slightly different initial data leads to a very different solution after a short time, meaning the equation lacks stability and is ill-posed.

3.11 Classification of 2nd Order Linear PDEs

We first consider the two variables case. If the unknown function u only depends on two independent variables (x,y) or (x,t), we classify the equations as follows:

(i) **Elliptic equation:** e.g. Laplace equation $u_{xx} + u_{yy} = 0$

(ii) **Hyperbolic equation:** e.g. wave equation $u_{tt} - a^2 u_{xx} = 0$

(iii) **Parabolic equation:** e.g. heat equation $u_t - k u_{xx} = 0$

Theorem 3.6 (classification of second-order linear PDEs). After a linear change of variables, any second-order linear PDE of the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = g(x,y)$$

can be reduced to one of three forms:

(i) Elliptic case: If $a_{12}^2 < a_{11}a_{22}$, it is reducible to $u_{xx} + u_{yy} + \cdots = 0$

(ii) Hyperbolic case: If $a_{12}^2 > a_{11}a_{22}$, it is reducible to $u_{xx} - u_{yy} + \cdots = 0$

(iii) Parabolic case: If $a_{12}^2 = a_{11}a_{22}$ (unless $a_{11} = a_{12} = a_{22} = 0$), it is reducible to $u_{xx} + \cdots = 0$

Classification can be done just by using the discriminant.

Proof. We focus on the second-order part of the PDE, since the classification is determined only by the coefficients of u_{xx}, u_{xy}, u_{yy}. We write the second-order part as $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy}$ and associate it with the symmetric matrix

$$\mathbf{A}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Then the second-order part may be regarded as the quadratic form

$$a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 = \begin{pmatrix} \xi & \eta \end{pmatrix} \mathbf{A} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Since $\mathbf{A}$ is a real symmetric matrix, it can be diagonalised by an orthogonal change of variables. Hence, there exists an orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the eigenvalues of $\mathbf{A}$. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Equivalently, (ξ, η) are new linear coordinates obtained from (x, y). Under this linear change of variables, the second-order part becomes

$$\lambda_1 u_{\xi\xi} + \lambda_2 u_{\eta\eta}.$$

The lower-order terms change into some new linear combination of u_ξ, u_η, u and the right-hand side becomes a function of (ξ, η). Thus they do not affect the classification.

We now classify according to the signs of λ_1 and λ_2. Observe that $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}^2$ so $\lambda_1\lambda_2 = \det(\mathbf{A}) = a_{11}a_{22} - a_{12}^2$. We now talk about the elliptic case as the hyperbolic and parabolic cases can be easily deduced once we have discussed the former. Suppose $a_{12}^2 < a_{11}a_{22}$. Then, $\det(\mathbf{A}) > 0$, so $\lambda_1\lambda_2 > 0$. As such, λ_1 and λ_2 have the same sign. Since we are assuming the second-order part is not identically zero, both eigenvalues are non-zero. Thus, after possibly multiplying the whole equation by -1, we may assume $\lambda_1 > 0$ and $\lambda_2 > 0$.

Now, rescale the coordinates by setting

$$X = \frac{\xi}{\sqrt{\lambda_1}} \quad \text{and} \quad Y = \frac{\eta}{\sqrt{\lambda_2}}$$

and one can deduce that $\lambda_1 u_{\xi\xi} + \lambda_2 u_{\eta\eta} = u_{XX} + u_{YY}$. Hence, the PDE is reduced to the form $u_{XX} + u_{YY} + \cdots = 0$, which is the elliptic case.

As such, one can eventually deduce that the classification depends only on the sign of $a_{12}^2 - a_{11}a_{22}$. □

As for the n variables case, consider m variables $x_1, \ldots, x_m$ and the equation

$$\sum_{i,j=1}^{m} a_{ij} u_{x_i x_j} + \sum_{i=1}^{m} b_i u_{x_i} + c_0 u = g(x_1, \ldots, x_m)$$

with real constants and symmetric coefficient matrix $\mathbf{A} = (a_{ij})$ having eigenvalues $d_1, \ldots, d_m$. Let $x = (x_1, \ldots, x_n)$ and fix $x_0 \in \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)$. Fix $x_0 \in \mathbb{R}^n$. The PDE is called

(i) **Elliptic** at x_0, if all the eigenvalues $d_1, \ldots, d_n$ are > 0 or < 0

(ii) **Hyperbolic** at x_0, if none of the eigenvalues vanish and one of them has the opposite sign from the $n-1$ others

(iii) **Parabolic** at x_0 if one of the eigenvalues is 0 and all the others have the same sign

Example 3.7. Find the region in the xy-plane where

$$y u_{xx} - 2u_{xy} + x u_{yy} = 0 \tag{3.10}$$

is elliptic, hyperbolic, or parabolic.

Solution. We compare (3.10) with the general second-order linear PDE

$$A u_{xx} + 2B u_{xy} + C u_{yy} = 0.$$

Thus, in this problem, $A = y$, $2B = -2$, and $C = x$. The discriminant is $B^2 - AC = 1 - xy$. By Theorem 3.6, (3.10) is hyperbolic when $xy < 1$, parabolic when $xy = 1$, and elliptic when $xy > 1$.[1] □

3.12 Solution to the Wave Equation on the Whole Line

Theorem 3.7 (solution to 1-dimensional wave equation). Consider the 1-

[1] Geometrically, the parabolic region is the hyperbola $xy = 1$. The equation is hyperbolic on the side where $xy < 1$, and elliptic on the side where $xy > 1$.

dimensional wave equation on the entire real line:

$$u_{tt} - c^2 u_{xx} = 0 \quad \text{for } x \in (-\infty, \infty).$$

The general solution is given by:

$$u(x,t) = f(x+ct) + g(x-ct),$$

where f and g are two arbitrary (twice differentiable) functions of a single variable.

We give two proofs — first using operator factorisation and second using a linear change of variables.

Proof. We can factor the wave operator as a difference of squares. That is,

$$(\partial_{tt} - c^2 \partial_{xx})u = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0.$$

Let $v = (\partial_t + c\partial_x)u = u_t + cu_x$. Then, the wave equation reduces to a first-order transport equation for v. That is, $v_t - cv_x = 0$. The solution to this transport equation is constant along the characteristic lines where $\frac{dx}{dt} = -c$. That is, $x = -ct + x_0$. Thus, $v(x,t) = h(x+ct)$ for some arbitrary function h. We now substitute v back to solve for u, thus obtaining

$$u_t + cu_x = h(x+ct).$$

We solve this using the method of characteristics. The characteristic curves satisfy $\frac{dx}{dt} = c$, so $x = ct - x_0$, where x_0 is a parameter. Along these characteristics, the total derivative of u with respect to time is:

$$\frac{d}{dt} u(ct - x_0, t) = cu_x + u_t = h(2ct - x_0).$$

Integrating with respect to t yields

$$u(ct - x_0, t) = \int h(2ct - x_0)\, dt = f(2ct - x_0) + g(x_0).$$

Substituting $x_0 = ct - x$, we obtain the general solution $u(x,t) = f(x+ct) + g(x-ct)$. □

We then give the second proof using a linear change of variables.

Proof. We introduce new variables $\xi = x + ct$ and $\eta = x - ct$. Applying the chain rule, one can show that the wave equation $u_{tt} - c^2 u_{xx} = 0$ transforms into $u_{\xi\eta} = 0$. Integrating with respect to η gives $u_\xi = h(\xi)$. Integrating again with respect to ξ gives

$$u = \int h(\xi)\, d\xi + g(\eta) = f(\xi) + g(\eta).$$

Substituting the original variables yields the general solution. □

Consider the initial value problem for the wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{if } t > 0 \text{ and } x \in \mathbb{R}; \\ u(x,0) = \phi(x) & \text{if } x \in \mathbb{R}; \\ u_t(x,0) = \psi(x) & \text{if } x \in \mathbb{R} \end{cases} \tag{3.11}$$

where ϕ and ψ are arbitrary initial conditions. The unique solution to this problem is given by d'Alembert's formula.

Theorem 3.8 (d'Alembert's formula). The solution to (3.11) is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds.$$

Proof. We know from Theorem 3.7 that the general solution is of the form

$$u(x,t) = f(x+ct) + g(x-ct).$$

At $t = 0$, applying the first initial condition yields $u(x,0) = \phi(x)$, so $f(x) + g(x) = \phi(x)$. Taking the time derivative of the general solution and then applying the second initial condition $u_t(x,0)$ at $t = 0$, we have $cf'(x) - cg'(x) = \psi(x)$. The result follows. □

Note that the solution to (3.11) can always be cleanly decoupled into two propagating waves $u(x,t) = F(x+ct) + G(x-ct)$, where

$$F(\xi) = \frac{1}{2}\phi(\xi) + \frac{1}{2c}\int_0^\xi \psi(s)\ ds \quad \text{and} \quad G(\xi) = \frac{1}{2}\phi(\xi) - \frac{1}{2c}\int_0^\xi \psi(s)\ ds$$

As shown in Figure 3.1, $F(x+ct)$ represents a left travelling wave (moving in the $-x$ direction with speed c), and $G(x-ct)$ represents a right travelling wave (moving in the $+x$ direction with speed c)[2].

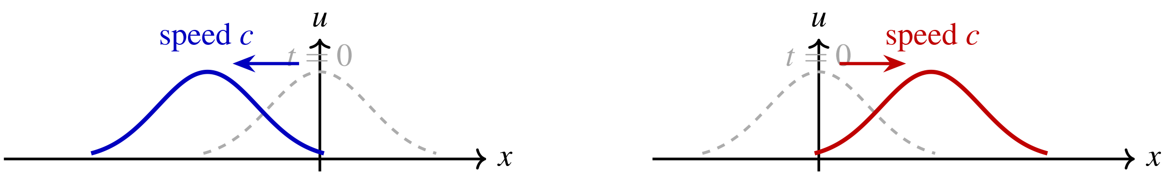


Figure 3.1: The solution of the one-dimensional wave equation

Now, we wish to solve the Dirichlet initial/boundary problem on the half-line $(0,\infty)$:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{if } x > 0 \text{ and } t > 0; \\ u(0,t) = 0 & \text{if } t \geq 0; \\ u(x,0) = \phi(x) & \text{if } x \geq 0; \\ u_t(x,0) = \psi(x) & \text{if } x \geq 0 \end{cases}$$

[2] If the initial profiles $\phi(x)$ and $\psi(x)$ are both odd (or both even) functions of x, then the resulting solution $u(x,t)$ is also an odd (respectively, even) function of x for all times $t > 0$.

The strategy involves extending the problem to the entire real line. First, we perform an odd extension. Since $u(0,t) = 0$, take odd extensions of the initial functions $\phi(x)$ and $\psi(x)$ to the whole line, denoted as $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$. That is,

$$\widetilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \geq 0; \\ -\phi(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad \widetilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } x \geq 0; \\ -\psi(-x) & \text{if } x < 0. \end{cases}$$

We then solve on the whole real line. Let $\widetilde{u}$ be the solution to the initial-value problem on $\mathbb{R}$ with data $\widetilde{\phi}$ and $\widetilde{\psi}$. The solution $\widetilde{u}$ is odd and can be derived via D'Alembert formula (Theorem 3.8). So, the solution $u(x,t)$ for $0 < x < \infty$ is the restriction of $\widetilde{u}$ to that domain.

For $x \geq 0$, the solution is given as follows:

(i) If $x \geq ct$,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

(ii) If $0 < x < ct$,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\, ds$$

Now, consider the inhomogeneous wave equation on the whole line. That is, to consider waves with a source term $f(x,t)$. So, the wave equation now becomes

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{if } t > 0 \text{ and } -\infty < x < \infty; \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

The solution is found by splitting u into $u^{(1)}$ (homogeneous with initial data) and $u^{(2)}$ (inhomogeneous with zero initial data). That is, we first solve the homogeneous problem $w_{tt} - c^2 w_{xx} = 0$ with $w(x,\tau) = 0$ and $w_t(x,\tau) = f(x,\tau)$, then note that the solution to the inhomogeneous problem with zero initial data is

$$u^{(2)}(x,t) = \int_0^t w(x,t;\tau)\, d\tau.$$

Recall that the 1-dimensional heat equation is $u_t = k u_{xx}$. Here, $u = u(x,t)$ may represent temperature or concentration at position x and time t, and k is the diffusion coefficient. Factorisation fails here, so qualitative properties are studied instead.

> Theorem 3.9 (weak maximum principle). If $u(x,t)$ satisfies the heat equation in the rectangle $Q = [0,l] \times [0,T]$, the maximum value of u is achieved either initially ($t = 0$) or on the lateral sides $x = 0$ or $x = l$).

See Figure 3.2 for an illustration of the weak maximum principle (Theorem 3.9).

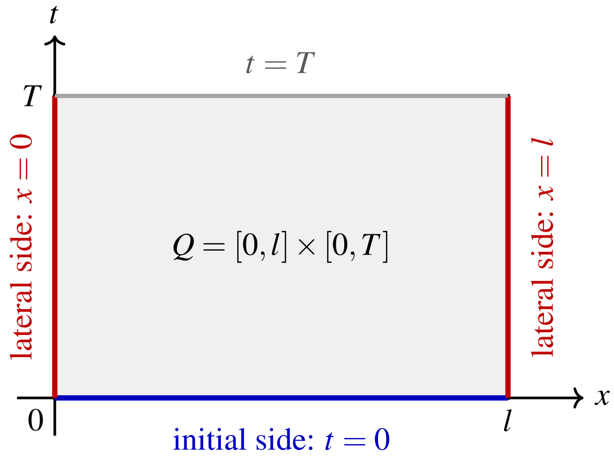


Figure 3.2: The weak maximum principle for the heat equation

Equivalently, if we define the parabolic boundary of Q by $\Gamma = ([0,l]\times\{0\})\cup(\{0,l\}\times[0,T])$, then the weak maximum principle states that

$$\max_Q u = \max_\Gamma u. \tag{3.12}$$

Let us see why an interior maximum gives a contradiction. Suppose first that a maximum is attained at an interior point (x_0,t_0) with $0<x_0<l$ and $0<t_0\le T$. At a spatial maximum at fixed time $t=t_0$, we have $u_x(x_0,t_0)=0$ and $u_{xx}(x_0,t_0)\le 0$. Also, if the maximum is first reached at time $t_0>0$, then intuitively $u_t(x_0,t_0)\ge 0$. But the heat equation gives

$$u_t(x_0,t_0)=ku_{xx}(x_0,t_0)\le 0$$

Thus one obtains the restrictive conditions $u_t(x_0,t_0)=0$ and $u_{xx}(x_0,t_0)=0$. This does not yet give a strict contradiction. Therefore, to prove the weak maximum principle rigorously, one perturbs the solution slightly.

Proof. Let $\varepsilon>0$ and define $v(x,t)=u(x,t)-\varepsilon t$. Then,

$$v_t = u_t-\varepsilon \quad \text{and} \quad v_{xx}=u_{xx}.$$

Since $u_t=ku_{xx}$, we have $v_t-kv_{xx}=-\varepsilon<0$. We claim that v cannot attain its maximum at an interior point of Q with $0<x<l$ and $0<t\le T$. Indeed, suppose v attains a maximum at such a point (x_0,t_0). At fixed time $t=t_0$, the function $x\mapsto v(x,t_0)$ has a local maximum at x_0, so

$$v_x(x_0,t_0)=0 \quad \text{and} \quad v_{xx}(x_0,t_0)\le 0.$$

Moreover, because $v(x_0,t_0)$ is maximal relative to earlier times, one has $v_t(x_0,t_0)\ge 0$. As such, $v_t(x_0,t_0)-kv_{xx}(x_0,t_0)\ge 0$, which contradicts $v_t-kv_{xx}=-\varepsilon<0$.

Hence the maximum of v is achieved on the parabolic boundary Γ. Since $v=u-\varepsilon t$, we have

$$\max_Q(u-\varepsilon t)=\max_\Gamma(u-\varepsilon t).$$

For every $(x,t) \in Q$, $0 \le t \le T$, so

$$u(x,t) - \varepsilon T \le u(x,t) - \varepsilon t \le u(x,t).$$

Therefore,

$$\max_Q u - \varepsilon T \le \max_Q (u - \varepsilon t) = \max_\Gamma (u - \varepsilon t) \le \max_\Gamma u.$$

Hence

$$\max_Q u \le \max_\Gamma u + \varepsilon T.$$

Letting $\varepsilon \to 0^+$ gives

$$\max_Q u \le \max_\Gamma u. \tag{3.13}$$

Since $\Gamma \subseteq Q$, the reverse inequality is automatic. Thus, (3.12) holds. □

We now state the corresponding n-dimensional result. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Define the parabolic cylinder $\Omega_T = \Omega \times (0,T]$, whose closure is $\overline{\Omega_T} = \overline{\Omega} \times [0,T]$. As shown in Figure 3.3, the parabolic boundary is

$$\partial_p \Omega_T = \left(\overline{\Omega} \times \{0\}\right) \cup \left(\partial\Omega \times [0,T]\right).$$

That is, the parabolic boundary consists of the initial time slice and the lateral spatial boundary.

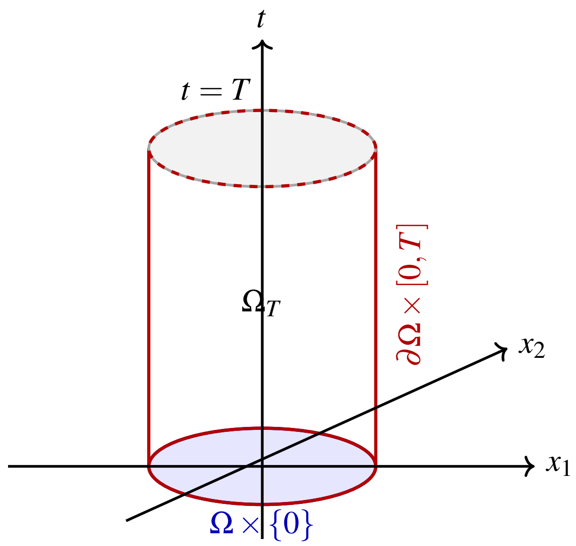


Figure 3.3: The parabolic cylinder Ω_T and its parabolic boundary

Theorem 3.10 (maximum principle in a bounded domain). Let $u : \overline{\Omega_T} \to \mathbb{R}$ be continuous and suppose $u_t - \Delta u \le 0$ in Ω_T. Then,

$$\max_{\overline{\Omega_T}} u = \max_{\partial_p \Omega_T} u.$$

A function satisfying $u_t - \Delta u \le 0$ is called a subsolution. A function satisfying $u_t - \Delta u \ge 0$ is called a supersolution.

The maximum principle also gives a minimum principle by applying it to $-u$.

Proposition 3.1 (minimum principle). If u satisfies $u_t - \Delta u \geq 0$ in Ω_T, then

$$\min_{\overline{\Omega_T}} u = \min_{\partial_p \Omega_T} u.$$

In particular, if a heat equation solution has zero initial value and zero boundary value, then both its maximum and its minimum are zero. Therefore the solution must vanish identically.

Theorem 3.11 (comparison principle). Let $u, v : \overline{\Omega_T} \to \mathbb{R}$ be continuous. Suppose $u_t - \Delta u \leq v_t - \Delta v$ in Ω_T and $u \leq v$ on $\partial_p \Omega_T$. Then, $u \leq v$ in Ω_T.

Proof. Let $w = u - v$. Then,

$$w_t - \Delta w = (u_t - \Delta u) - (v_t - \Delta v) \leq 0 \quad \text{in } \Omega_T. \tag{3.14}$$

Also, on the parabolic boundary, $w = u - v \leq 0$. By the maximum principle (Theorem 3.10),

$$\max_{\overline{\Omega_T}} w = \max_{\partial_p \Omega_T} w \leq 0. \tag{3.15}$$

Hence $w \leq 0$ in Ω_T, which means $u \leq v$ in Ω_T. □

The bounded-domain maximum principle (Theorem 3.10) assumes that the spatial domain is bounded. A version is also true in the full space $\mathbb{R}^n$, provided the solution is bounded.

Theorem 3.12 (maximum principle in full space). Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a bounded continuous function satisfying $u_t - \Delta u \leq 0$ for all $t \in (0,T]$. Then,

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{x \in \mathbb{R}^n} u(x,0).$$

Proof. The idea is to compare u with a supersolution that grows quadratically as $|x| \to \infty$. Let $\varepsilon > 0$ and define

$$v(x,t) = \varepsilon \left(t + \frac{|x|^2}{2n} \right) + \sup_{y \in \mathbb{R}^n} u(y,0).$$

Then, $v_t = \varepsilon$ and $\Delta v = \varepsilon$. Hence, $v_t - \Delta_v = 0$. Also, at $t = 0$,

$$v(x,0) = \varepsilon \cdot \frac{|x|^2}{2n} + \sup_{y \in \mathbb{R}^n} u(y,0) \geq u(x,0).$$

Since u is bounded, choose $M > 0$ such that $|u(x,t)| \leq M$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. On the lateral boundary $\partial B_R \times [0,T]$, we have

$$v(x,t) \geq \varepsilon \cdot \frac{R^2}{2n} + \sup_{y \in \mathbb{R}^n} u(y,0). \tag{3.16}$$

For R sufficiently large, this is greater than M, and hence greater than $u(x,t)$ on $\partial B_R \times [0,T]$. Now apply the comparison principle in $B_R \times (0,T]$ (Theorem 3.11). Since $u \le v$ on the parabolic boundary, we get

$$u(x,t) \le v(x,t) \quad \text{for } (x,t) \in B_R \times [0,T].$$

Letting $R \to \infty$, this inequality holds for any fixed $(x,t) \in \mathbb{R}^n \times [0,T]$. Therefore

$$u(x,t) \le \varepsilon \left(t + \frac{|x|^2}{2n}\right) + \sup_{y\in\mathbb{R}^n} u(y,0).$$

Finally, letting $\varepsilon \to 0^+$ gives

$$u(x,t) \le \sup_{y\in\mathbb{R}^n} u(y,0).$$

Taking the supremum over (x,t) gives

$$\sup_{\mathbb{R}^n\times[0,T]} u \le \sup_{x\in\mathbb{R}^n} u(x,0).$$

The opposite inequality is immediate because $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^n \times [0,T]$. Hence equality holds. □

3.13 Fundamental Solution of the Heat Equation

Now, we discuss the uniqueness of the heat equation. Consider the Dirichlet boundary value problem as follows:

$$\begin{cases} u_t = ku_{xx} & \text{if } x \in [0,l] \text{ and } t > 0; \\ u(x,0) = \phi(x) & \text{if } x \in [0,l]; \\ u(0,t) = g(t) & \text{if } t \ge 0; \\ u(l,t) = h(t) & \text{if } t \ge 0. \end{cases} \tag{3.17}$$

Theorem 3.13 (uniqueness theorem). There is at most one solution to (3.17).

We give two proofs to Theorem 3.13 — first by the maximum principle and second by the energy method.

Proof. We first discuss the proof using the maximum principle. Suppose u_1 and u_2 are both solutions with the same initial and boundary data. Define

$$w(x,t) = u_1(x,t) - u_2(x,t).$$

Then, w satisfies

$$\begin{cases} w_t = kw_{xx} & x \in [0,l] \text{ and } t > 0; \\ w(x,0) = 0 & x \in [0,l]; \\ w(0,t) = 0 & t \ge 0; \\ w(l,t) = 0 & t \ge 0. \end{cases}$$

Fix $\overline{T} > 0$. Apply the maximum principle to w on $[0,l] \times [0,\overline{T}]$ (Theorem 3.10). Since $w = 0$ on the parabolic boundary, then

$$\max_{[0,l]\times[0,\overline{T}]} w = 0.$$

Apply the minimum principle to w (Proposition 3.1). Again using $w = 0$ on the parabolic boundary,

$$\min_{[0,l]\times[0,\overline{T}]} w = 0. \tag{3.18}$$

Thus $w = 0$ on $[0,l] \times [0,\overline{T}]$. Since $\overline{T} > 0$ was arbitrary, we obtain $w = 0$ on $[0,l] \times [0,\infty)$. Therefore $u_1 = u_2$, proving uniqueness. □

We then give a proof using the energy method (Lemma 3.1). The energy method proves uniqueness by showing that the square integral of the difference of two solutions cannot increase.

Lemma 3.1 (energy monotonicity). Let $\Omega \subseteq \mathbb{R}$ be a bounded open set with piecewise C^1 boundary. Suppose w satisfies

$$\begin{cases} w_t = kw_{xx} & \text{if } x \in \Omega \text{ and } t > 0; \\ w(x,0) = 0 & \text{if } x \in \Omega, \end{cases}$$

and suppose either $w(x,t) = 0$ for $x \in \partial\Omega$ for the Dirichlet boundary condition, or $w_\nu(x,t) = 0$ for $x \in \partial\Omega$ for the Neumann boundary condition. Define

$$E(t) = \int_\Omega w(x,t)^2 \, dx.$$

Then $E(t)$ is non-increasing in time.

We first give a proof of Lemma 3.1.

Proof. Differentiating $E(t)$ yields

$$E'(t) = 2\int_\Omega ww_t \, dx = 2k\int_\Omega ww_{xx} \, dx.$$

Integrating by parts gives

$$\int_\Omega ww_{xx} \, \mathrm{dx} = [ww_x]_{\partial\Omega} - \int_\Omega w_x^2 \, \mathrm{dx}.$$

Under the Dirichlet condition $w = 0$ on $\partial\Omega$, the boundary term vanishes. Under the Neumann condition $w_\nu = 0$ on $\partial\Omega$, the boundary term also vanishes. Therefore

$$E'(t) = -2k\int_\Omega w_x^2 \, dx \leq 0.$$

Thus $E(t)$ is non-increasing. □

We now prove Theorem 3.13 using Lemma 3.1.

Proof. If $w(x,0)=0$, then

$$E(0)=\int_\Omega w(x,0)^2\,dx=0.$$

Since $E(t)$ is non-increasing and non-negative, we get $0\le E(t)\le E(0)=0$. Hence $E(t)=0$ for all t, and therefore $w(x,t)=0$ for all x and t. This gives uniqueness. □

Uniqueness says that the solution is determined by its data. Stability says that small changes in the data lead to small changes in the solution. Consider two solutions u_1 and u_2 of the Dirichlet problem with possibly different initial and boundary data for $i=1,2$:

$$\begin{cases} (u_i)_t = k(u_i)_{xx} & \text{if } x\in[0,l] \text{ and } t>0; \\ u_i(x,0)=\phi_i(x) & \text{if } x\in[0,l]; \\ u_i(0,t)=g_i(t) & \text{if } t\ge 0; \\ u_i(l,t)=h_i(t) & \text{if } t\ge 0. \end{cases}$$

Let $w=u_1-u_2$. Then,

$$\begin{cases} w_t = kw_{xx} & \text{if } x\in[0,l] \text{ and } t>0; \\ w(x,0)=\phi_1(x)-\phi_2(x) & \text{if } x\in[0,l]; \\ w(0,t)=g_1(t)-g_2(t) & \text{if } t\ge 0; \\ w(l,t)=h_1(t)-h_2(t) & \text{if } t\ge 0. \end{cases}$$

Applying the maximum principle to w and $-w$ gives

$$|w(x,t)|\le \max\left\{\max_{x\in[0,l]}|\phi_1(x)-\phi_2(x)|,\ \max_{s\in[0,t]}|g_1(s)-g_2(s)|,\ \max_{s\in[0,t]}|h_1(s)-h_2(s)|\right\}.$$

Thus, if the initial and boundary data are uniformly close, then the solutions are uniformly close.

Now suppose the boundary condition is fixed, either Dirichlet or Neumann, and only the initial data varies. Let u_1,u_2 solve the heat equation with initial data ϕ_1,ϕ_2. Define $w=u_1-u_2$. Then w satisfies the same homogeneous boundary condition and has initial data

$$w(x,0)=\phi_1(x)-\phi_2(x).$$

Define

$$E(t)=\int_\Omega w(x,t)^2\,dx.$$

By the energy monotonicity lemma (Lemma 3.1), $E(t)\le E(0)$. Therefore

$$\int_\Omega (u_1(x,t)-u_2(x,t))^2\,dx \le \int_\Omega (\phi_1(x)-\phi_2(x))^2\,dx. \tag{3.19}$$

This is stability in the square-integral, or L^2, sense.

We now solve the initial value problem

$$\begin{cases} u_t = ku_{xx} & \text{if } t > 0 \text{ and } -\infty < x < \infty; \\ u(x,0) = \phi(x) & \text{if } -\infty < x < \infty. \end{cases}$$

The idea is to first find the solution corresponding to a single spike of unit mass at the origin. This special solution is called the fundamental solution, source function, or heat kernel. Then, by translating and superposing such solutions, one obtains the solution for general initial data.

Proposition 3.2. Assume u is a solution of the heat equation $u_t = k\Delta u$ in $\mathbb{R}^n \times (0,\infty)$. The heat equation has several important invariance properties.

- **(i) Linearity:** If u and v are solutions, then for constants $a,b \in \mathbb{R}$, $au+bv$ is also a solution.
- **(ii) Translation:** If $u(x,t)$ is a solution, then $v(x,t) = u(x-y,t)$ is also a solution for each fixed $y \in \mathbb{R}^n$.
- **(iii) Multiplication by constant:** If $u(x,t)$ is a solution and $C \in \mathbb{R}$, then $v(x,t) = Cu(x,t)$ is also a solution.
- **(iv) Derivatives of solutions:** If u is a sufficiently smooth solution, then any derivative of u, such as u_x, u_t, or u_{xx}, is again a solution.

Proof. These are easy to verify. □

Example 3.8. For example, in one dimension, let $v = u_x$. Then,

$$v_t = (u_x)_t = (u_t)_x = (ku_{xx})_x = k(u_x)_{xx} = kv_{xx}.$$

The heat equation has a distinctive parabolic scaling. Suppose $u(x,t)$ solves $u_t = ku_{xx}$. For $b \in \mathbb{R}$, define $\widetilde{u}(x,t) = u(bx, b^2t)$. Then $\widetilde{u}$ also solves the heat equation. This is easy to see. So, the natural scaling of the heat equation is

$$x \mapsto bx \quad \text{and} \quad t \mapsto b^2 t.$$

This means that time scales like length squared.

Next, we discuss rotation invariance. In $\mathbb{R}^n$, the heat equation $u_t = k\Delta u$ is invariant under rotations. Let $\mathbf{Q}$ be an orthogonal matrix, so $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$. If $u(x,t)$ solves the heat equation, define

$$v(x,t) = u(Qx,t).$$

Then v is also a solution. The reason is that the Laplacian is rotation invariant. That is,

$$\Delta_x v(x,t) = \Delta_y u(y,t) \quad \text{where } y = \mathbf{Q}x.$$

As such,

$$v_t(x,t) = u_t(\mathbf{Q}x,t) = k\Delta_y u(\mathbf{Q}x,t) = k\Delta_x v(x,t).$$

We now give a heuristic idea for solving the heat equation. The initial data $\phi(x)$ can be viewed heuristically as a continuous superposition of point masses. That is,

$$\phi(x) \sim \int_{-\infty}^{\infty} \phi(y)\delta(x-y)\,dy.$$

Thus the strategy is to first solve the heat equation with initial data $\delta(x)$, a unit spike at the origin. Then, translate this solution to obtain the solution with initial data $\delta(x-y)$. Lastly, superpose all translated solutions with weights $\phi(y)$. This leads to a solution of the form

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy,$$

where $S(x,t)$ is the fundamental solution.

Definition 3.2 (fundamental solution). The fundamental solution of the heat equation is the solution $S(x,t)$ satisfying

$$S_t = kS_{xx} \quad \text{and} \quad S(x,0) = \delta(x).$$

It is also called the source function or heat kernel.

The initial condition $S(x,0) = \delta(x)$ in Definition 3.2 is not a classical pointwise condition. It is understood in the sense of distributions. Instead of solving directly for S, we first solve an initial value problem with Heaviside initial data:

$$\begin{cases} Q_t = kQ_{xx} & \text{if } t > 0 \text{ and } x \in \mathbb{R}; \\ Q(x,0) = H(x) & \end{cases}$$

Then, the derivative

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t) \quad \text{satisfies} \quad S(x,0) = H'(x) = \delta(x).$$

We seek a self-similar solution of the form

$$Q(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right).$$

Let $p = \frac{x}{\sqrt{4kt}}$. Then, $Q(x,t) = g(p)$. We compute the derivatives. First,

$$p_x = \frac{1}{\sqrt{4kt}}.$$

Hence,

$$Q_x = g'(p)\,p_x = \frac{1}{\sqrt{4kt}} g'(p).$$

Differentiating again,

$$Q_{xx} = \frac{1}{4kt} g''(p).$$

Next, we have

$$p_t = -\frac{x}{2t\sqrt{4kt}} = -\frac{p}{2t}.$$

Therefore,

$$Q_t = g'(p)p_t = -\frac{p}{2t}g'(p).$$

Substituting into the heat equation $Q_t = kQ_{xx}$ yields

$$-\frac{p}{2t}g'(p) = k\cdot\frac{1}{4kt}g''(p).$$

Hence,

$$-\frac{p}{2t}g'(p) = \frac{1}{4t}g''(p).$$

Multiplying by $4t$, we obtain $-2pg'(p) = g''(p)$ or equivalently, $g''(p)+2pg'(p) = 0$. Let $h(p) = g'(p)$. Then, $h'(p)+2ph(p) = 0$. One can solve this ODE and eventually obtain

$$g(p) = C_1\int_0^p e^{-s^2}\,ds + C_2.$$

Thus,

$$Q(x,t) = C_1\int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds + C_2.$$

We determine C_1 and C_2 using the initial condition $Q(x,0) = H(x)$. If $x > 0$, then as $t \to 0^+$, $\frac{x}{\sqrt{4kt}} \to +\infty$ so Hence

$$Q(x,0^+) = C_1\int_0^{\infty} e^{-s^2}\,ds + C_2 = 1.$$

If $x < 0$, then as $t \to 0^+$, $\frac{x}{\sqrt{4kt}} \to -\infty$ so

$$Q(x,0^+) = C_1\int_0^{-\infty} e^{-s^2}\,ds + C_2 = 0.$$

Recall that

$$\int_0^{\infty} e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2} \quad \text{so} \quad \int_0^{-\infty} e^{-s^2}\,s = -\frac{\sqrt{\pi}}{2}.$$

Hence, one can deduce that $C_1 = \frac{1}{\sqrt{\pi}}$ and $C_2 = \frac{1}{2}$. Therefore,

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-p^2}\,dp.$$

As such, the fundamental solution is $S(x,t) = Q_x(x,t)$. This implies that

$$S(x,t) = \frac{1}{2\sqrt{\pi kt}}e^{-x^2/(4kt)}.$$

We can also write

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}. \tag{3.20}$$

This is the heat kernel on the whole line (Figure 3.4).

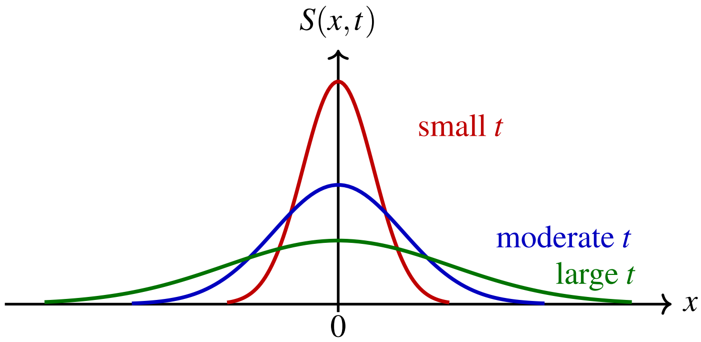


Figure 3.4: The heat kernel spreads out and decreases in height as t increases

We now use the fundamental solution to solve the heat equation with general initial data.

Theorem 3.14 (solution formula for the heat equation on the whole line). The solution to the heat equation

$$\begin{cases} u_t = ku_{xx} & \text{if } t > 0 \text{ and } x \in \mathbb{R}; \\ u(x,0) = \phi(x) & \text{if } x \in \mathbb{R} \end{cases}$$

is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad \text{where} \quad S(k,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

Equivalently,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)} \phi(y)\,dy.$$

Proof. Define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,cy.$$

First, we show that u satisfies the heat equation. Since $S_t = kS_{xx}$, we have, formally differentiating under the integral sign,

$$u_t(x,t) = \int_{-\infty}^{\infty} S_t(x-y,t)\phi(y)\,dy = k\frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy = ku_{xx}(x,t).$$

Next, we verify the initial condition. Since $S(x,t) \to \delta(x)$ as $t \to 0^+$ in the sense of distributions, then

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \to \int_{-\infty}^{\infty} \delta(x-y)\phi(y)\,dy = \phi(x).$$

Hence $u(x,0) = \phi(x)$. □

Recall from (3.20) that the heat kernel is

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}.$$

We make some observations.

Proposition 3.3. The heat kernel (3.20) satisfies the following properties:

(i) Mass conservation:

$$\int_{-\infty}^{\infty} S(x,t)\,dx = 1$$

(ii) Self-similarity: The heat kernel can be written in the form

$$S(x,t) = \frac{1}{\sqrt{t}}\Phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{where} \quad \Phi(\xi) = \frac{1}{\sqrt{4\pi k}}e^{-\xi^2/(4k)}.$$

This reflects the parabolic scaling $x \sim \sqrt{t}$.

(iii) Instantaneous smoothing: For any $t > 0$, the function $S(x,t)$ is smooth in x. Therefore, if

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy,$$

then u becomes smooth immediately for $t > 0$, even if ϕ is not smooth. More precisely,

$$\partial_x^m u(x,t) = \int_{-\infty}^{\infty} \partial_x^m S(x-y,t)\phi(y)\,dy$$

for $t > 0$, under suitable integrability assumptions.

(iv) Decay: We have

$$\max_{x\in\mathbb{R}} S(x,t) \to 0 \quad \text{as } t \to \infty$$

Proof. **(i)** is easy to verify as one can use the Gaussian integral to prove that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\,dx = 1.$$

Thus the heat kernel has total mass one for every $t > 0$.

Next, we prove **(iv)**. This follows from the fact that

$$\max_{x\in\mathbb{R}} S(x,t) = S(0,t) = \frac{1}{\sqrt{4\pi kt}}.$$

Moreover, if $\phi \in L^1(\mathbb{R})$, then

$$|u(x,t)| \le \int_{-\infty}^{\infty} S(x-y,t)|\phi(y)|\,dy \le \max_{z\in\mathbb{R}} S(z,t)\int_{-\infty}^{\infty} |\phi(y)|\,dy = \frac{1}{\sqrt{4\pi kt}}\|\phi\|_{L^1(\mathbb{R})}.$$

Thus,

$$\|u(\cdot,t)\|_{L^\infty} \le \frac{1}{\sqrt{4\pi kt}}\|\phi\|_{L^1}.$$

This implies $u(x,t) \to 0$ uniformly in x at rate $t^{-1/2}$, provided $\phi \in L^1(\mathbb{R})$. □

3.14 Comparison Between Waves and Diffusions

The wave equation and the heat equation have very different qualitative behaviours. For the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$, recall that the solution on the whole line is given by d'Alembert's formula (Theorem 3.8), which is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy.$$

In particular, information propagates with finite speed c. The value at (x,t) only depends on the initial data inside the interval $[x-ct, x+ct]$.

On the other hand, for the heat equation $u_t = ku_{xx}$ on the whole line, the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \quad \text{where} \quad S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}.$$

Since $S(x-y,t) > 0$ for every $x,y \in \mathbb{R}$ and every $t > 0$, the solution at any point x is influenced immediately by the initial data at every point y. This is infinite speed of propagation.

Property	**Waves**	**Diffusions**
Speed of propagation	Finite speed, bounded by c	Infinite speed
Singularities for $t > 0$	Transported along characteristics	Lost immediately
Well-posed for $t > 0$	Yes	Yes, at least for bounded solutions
Well-posed for $t < 0$	Yes	No
Maximum principle	No	Yes
Behaviour as $t \to +\infty$	Energy is constant and does not decay	Solution decays to zero if the initial data is integrable
Information	Transported	Lost gradually

Table 3.1: Comparison between waves and diffusions.

For the wave equation, the domain of dependence is a cone. The point (x,t) only depends on the initial data from $x-ct$ to $x+ct$ (Figure 3.5).

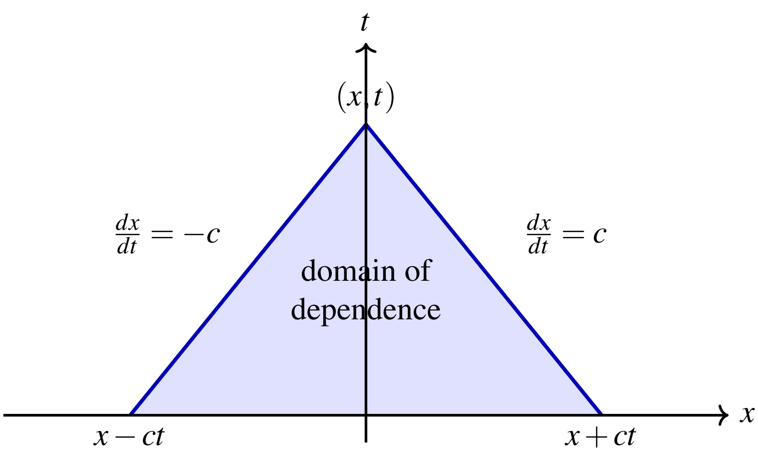


Figure 3.5: Finite speed of propagation for the wave equation

For the heat equation,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)} \phi(y)\, dy.$$

If $\phi \geq 0$, $\phi \not\equiv 0$, and $t > 0$, then $u(x,t) > 0$ for every $x \in \mathbb{R}$. Thus even if the initial heat is concentrated in a small region, the temperature becomes positive everywhere immediately (Figure 3.6).

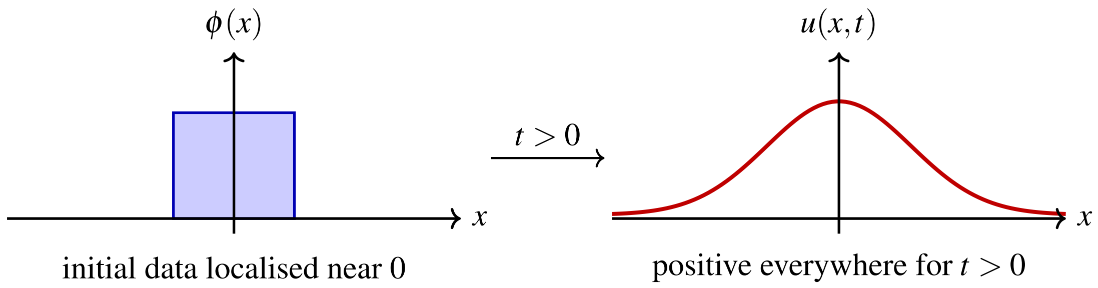


Figure 3.6: Infinite speed of propagation for the heat equation

Next, the heat equation satisfies a maximum principle, but the wave equation does not. Consider the function

$$u(x,t) = -x^2 - (t-1)^2.$$

Then, $u_{tt} = -2$ and $u_{xx} = -2$. Thus, $u_{tt} = u_{xx}$ so u solves the wave equation with $c = 1$.

Consider the domain $-1 < x < 1$ and $t > 0$. The value of u is maximised at $(x,t) = (0,1)$ where $u(0,1) = 0$. This maximum occurs inside the domain, not on the boundary (Figure 3.7). Therefore, the maximum principle does not hold for the wave equation.

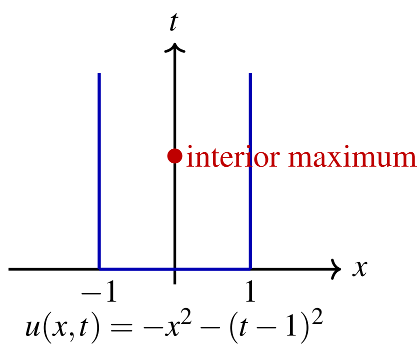


Figure 3.7: A wave-equation solution may attain its maximum in the interior

CHAPTER 4

Boundary Value Problems

4.1 Dirichlet Boundary Problem for the Wave Equation

We now solve the boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{if } t > 0 \text{ and } 0 < x < l; \\ u(0,t) = 0 = u(l,t) & \text{if } t \geq 0; \\ u(x,0) = \phi(x) & \text{if } 0 \leq x \leq l; \\ u_t(x,0) = \psi(x) & \text{if } 0 \leq x \leq l. \end{cases} \tag{4.1}$$

The key idea is to build the general solution as a linear combination of special solutions, called separated solutions. That is to say, we seek solutions of the form

$$u(x,t) = X(x)T(t).$$

Substituting into the wave equation gives

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Assuming $X(x)T(t) \neq 0$, we divide by $c^2 X(x)T(t)$ to obtain

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}.$$

The left-hand side depends only on t, while the right-hand side depends only on x. Hence both sides must be a constant. Let

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = a.$$

Thus we get

$$X''(x) = aX(x) \quad \text{and} \quad T''(t) = ac^2 T(t).$$

The boundary conditions give $X(0)=0$ and $X(l)=0$. Therefore we must solve the boundary value problem

$$\begin{cases} X''(x)-aX(x)=0 & \text{if } 0<x<l; \\ X(0)=0=X(l). \end{cases}$$

We first solve the spatial boundary value problem. We discuss three cases.

- **Case 1:** Suppose $a>0$. Then let $a=\alpha^2$, where $\alpha>0$. Then,

  $$X''-\alpha^2X=0.$$

  From MA3220 Ordinary Differential Equations, we know that the general solution is

  $$X(x)=C_1e^{\alpha x}+C_2e^{-\alpha x}.$$

  The conditions $X(0)=0$ and $X(l)=0$ imply $C_2=-C_1$ and $C_1\left(e^{\alpha l}-e^{-\alpha l}\right)=0$ respectively. Since $e^{\alpha l}-e^{-\alpha l}\neq 0$, we have $C_1=0$ so $C_2=0$. Hence, only the trivial solution exists. That is, $X=0$.

- **Case 2:** Suppose $a=0$. Then, $X''=0$. Then, the general solution is

  $$X(x)=C_1x+C_2.$$

  The boundary conditions yield $C_1=C_2=0$. Again, $X=0$.

- **Case 3:** Suppose $a<0$. Let $a=-\beta^2$, where $\beta>0$. Then,

  $$X''+\beta^2X=0.$$

  The general solution is

  $$X(x)=C_1\cos(\beta x)+C_2\sin(\beta x).$$

  The condition $X(0)=0$ gives $C_1=0$, whereas $X(l)=0$ gives $C_2\sin(\beta l)=0$. For a non-trivial solution, we require $C_2\neq 0$ so $\sin(\beta l)=0$. As such, there exists $n\in\mathbb{N}$ such that $\beta l=n\pi$. Hence, the non-trivial separated solutions occur precisely when

  $$a=-\left(\frac{n\pi}{l}\right)^2 \quad \text{where } n\in\mathbb{N}.$$

  The corresponding spatial functions are

  $$X_n(x)=\sin\left(\frac{n\pi x}{l}\right).$$

We then solve the time equation for the wave equation. Note that for each $\beta=\frac{n\pi}{l}$, we have $a=-\beta^2$. As the time equation is

$$T''(t)=ac^2T(t),$$

then

$$T''(t)+\beta^2c^2T(t)=0.$$

The general solution is

$$T_n(t) = A_n \cos(\beta ct) + B_n \sin(\beta ct).$$

So,

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right).$$

Multiplying $X_n(x)$ and $T_n(t)$, we obtain separated solutions

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right).$$

For each $n \in \mathbb{N}$, u_n solves the wave equation and satisfies zero Dirichlet boundary conditions.

Since the wave equation is homogeneous and linear, any finite sum of the separated solutions is again a solution. We therefore expect the general solution to be an infinite linear combination:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right).$$

Formally, to satisfy the initial condition $u(x,0) = \phi(x)$, we need

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

Thus the coefficients A_n are the Fourier sine coefficients of ϕ:

$$A_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

Next,

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-A_n \frac{n\pi c}{l} \sin\left(\frac{n\pi ct}{l}\right) + B_n \frac{n\pi c}{l} \cos\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right).$$

At $t = 0$, this gives

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right).$$

To have $u_t(x,0) = \psi(x)$, we need

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin\left(\frac{n\pi x}{l}\right).$$

Thus,

$$B_n \frac{n\pi c}{l} = \frac{2}{l}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx \quad \text{so} \quad B_n = \frac{2}{n\pi c}\int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

Therefore, the formal solution of the Dirichlet wave problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right),$$

where

$$A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \quad \text{and} \quad B_n = \frac{2}{n\pi c}\int_0^l \psi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

The functions

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \quad \text{where } n = 1,2,3,\ldots$$

are the spatial modes satisfying zero Dirichlet boundary conditions. These are called the sine eigenfunctions (Figure 4.1).

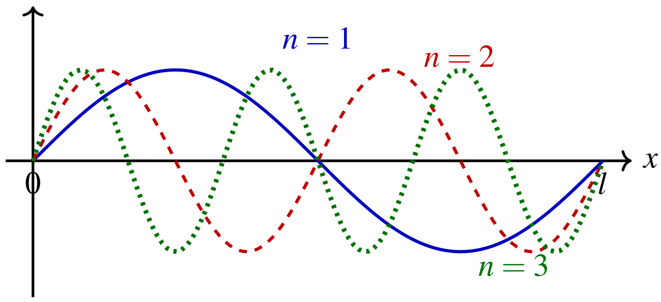


Figure 4.1: The first few sine modes satisfying $X(0) = X(l) = 0$

4.2 Dirichlet Boundary Problem for the Heat Equation

We now solve the heat equation with zero Dirichlet boundary conditions:

$$\begin{cases} u_t = ku_{xx} & \text{if } t > 0 \text{ and } 0 < x < l; \\ u(0,t) = 0 = u(l,t) & \text{if } t \geq 0; \\ u(x,0) = \phi(x) & \text{if } 0 \leq x \leq l. \end{cases} \tag{4.2}$$

Again, we seek a separated solution

$$u(x,t) = X(x)T(t).$$

Substituting into the PDE gives

$$X(x)T'(t) = kX''(x)T(t).$$

Dividing by $kX(x)T(t)$, we obtain

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = a.$$

Thus,

$$X''(x) = aX(x) \quad \text{and} \quad T'(t) = kaT(t).$$

The boundary conditions give $X(0) = 0$ and $X(l) = 0$. Thus, the spatial boundary value problem is the same as before:

$$\begin{cases} X'' - aX = 0 \\ X(0) = 0 = X(l) \end{cases}$$

From the previous analysis in Chapter 4.1, non-trivial solutions occur only when

$$a = -\left(\frac{n\pi}{l}\right)^2 \quad \text{where } n \in \mathbb{N}.$$

The corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{l}\right).$$

Now solve the time equation:

$$T'(t) = kaT(t).$$

With $a = -\left(\frac{n\pi}{l}\right)^2$, we obtain

$$T_n'(t) = -k\left(\frac{n\pi}{l}\right)^2 T_n(t).$$

Hence,

$$T_n(t) = C_n e^{-k(n\pi/l)^2 t}.$$

Therefore the separated solutions are

$$u_n(x,t) = C_n e^{-k(n\pi/l)^2 t} \sin\left(\frac{n\pi x}{l}\right).$$

By superposition, the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k(n\pi/l)^2 t} \sin\left(\frac{n\pi x}{l}\right).$$

To satisfy $u(x,0) = \phi(x)$, we require

$$\phi(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right).$$

Thus C_n are the Fourier sine coefficients of ϕ:

$$C_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

As such, the formal solution of the zero Dirichlet heat problem is

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-k(n\pi/l)^2 t} \sin\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad C_n = \frac{2}{l}\int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

There remains a natural question:

> Why should we look for solutions of the form $u(x,t) = X(x)T(t)$?

The reason is analogous to solving linear systems of ordinary differential equations. Consider the ODE system

$$u'(t) = \mathbf{A}u(t) \quad \text{where} \quad u : \mathbb{R} \to \mathbb{R}^n \text{ and } \mathbf{A} \in \mathcal{M}_{n\times n}(F).$$

Here, we take F to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. If $\mathbf{A}$ has an eigenvalue λ with eigenvector $\mathbf{v}$, so that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, then $u(t)\, e^{\lambda t}\mathbf{v}$ is a solution[1].

[1] Easy proof as learnt from MA3220 Ordinary Differential Equations.

For PDEs, the same idea appears with spatial differential operators. For example, write the heat equation as $u_t = Lu$, where $L = k\frac{\partial^2}{\partial x^2}$. We look for eigenfunctions X of L. That is,

$$LX = \lambda X.$$

If $u(x,t) = T(t)X(x)$, then substituting into $u_t = Lu$ gives

$$T'(t)X(x) = T(t)LX(x) = T(t)\lambda X(x).$$

Cancelling $X(x)$, we obtain the ODE

$$T'(t) = \lambda T(t).$$

Thus the PDE reduces to an eigenvalue problem in x and an ODE in t. We infer that the method of separation of variables is essentially the infinite-dimensional analogue of diagonalising a matrix.

4.3 Comparison Between Separated Wave and Heat Solutions

Both the wave equation and the heat equation with zero Dirichlet boundary conditions use the same spatial eigenfunctions:

$$X_n(x) = \sin(\beta x) \quad \text{where } \beta = \frac{n\pi}{l}.$$

However, their time factors are very different. For the wave equation,

$$T_n(t) = A_n \cos(\beta ct) + B_n \sin(\beta ct).$$

These are oscillatory functions. They do not decay in time. On the other hand, for the heat equation,

$$T_n(t) = C_n e^{-k\beta^2 t}.$$

These decay exponentially as $t \to \infty$. Moreover, higher-frequency modes decay faster because $\beta^2 = (n\pi/l)^2$ increases with n.

Feature	**Wave equation**	**Heat equation**
Spatial modes	$\sin\left(\frac{n\pi x}{l}\right)$	$\sin\left(\frac{n\pi x}{l}\right)$
Time factor	$\cos\left(\frac{n\pi ct}{l}\right)$, $\sin\left(\frac{n\pi ct}{l}\right)$	$e^{-k(n\pi/l)^2 t}$
Time behaviour	Oscillatory	Exponentially decaying
High-frequency modes	Oscillate faster	Decay faster
Energy	Conserved	Decays

Table 4.1: Comparison of separated solutions for wave and heat equations

4.4 General Eigenvalue Problem Viewpoint

Let L be a spatial differential operator. We may think of PDEs such as $u_t = Lu$ or $u_{tt} = Lu$ as infinite-dimensional analogues of ODE systems. To find separated solutions, we solve the eigenvalue problem

$$LX = \lambda X$$

together with the boundary conditions.

For example, with zero Dirichlet boundary conditions on $[0,l]$, one solves

$$\begin{cases} -X'' = \lambda X & \text{if } 0 < x < l; \\ X(0) = 0 = X(l). \end{cases}$$

This produces

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right).$$

For each eigenfunction X_n, the PDE reduces to an ODE for the time factor $T_n(t)$.

4.5 The Wave Equation with Neumann Boundary Conditions

Now consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{if } t > 0 \text{ and } 0 < x < l; \\ u_x(0,t) = u_x(l,t) = 0. \end{cases}$$

Again, seek separated solutions

$$u(x,t) = X(x)T(t).$$

Substituting into the PDE, we have

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing by $c^2 X(x)T(t)$, we obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = -\lambda.$$

One can deduce that the general separated-series solution is

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi c}{l} t\right) + B_n \sin\left(\frac{n\pi c}{l} t\right)\right] \cos\left(\frac{n\pi x}{l}\right).$$

For the Neumann wave equation, the $n = 0$ mode gives the term $A_0 + B_0 t$. This corresponds to the spatially constant mode. Unlike the higher modes, it does not oscillate in space.

4.6 Neumann Boundary Problem for the Heat Equation

We now consider the heat equation with Neumann boundary conditions:

$$\begin{cases} u_t = ku_{xx} & \text{if } t > 0 \text{ and } 0 < x < l; \\ u_x(0,t) = 0 = u_x(l,t) & \text{if } t \geq 0. \end{cases} \tag{4.3}$$

The Neumann boundary condition means that the spatial derivative vanishes at the boundary. Physically, it corresponds to no heat flux through the endpoints.

Say we seek a separated solution

$$u(x,t) = X(x)T(t).$$

Repeating the usual steps as in Chapter 4.2 yields

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda.$$

By applying the Neumann boundary conditions $X'(0) = 0$ and $X'(l) = 0$, the spatial eigenvalue problem is

$$\begin{cases} -X'' = \lambda X & \text{if } 0 < x < l; \\ X'(0) = 0 = X'(l). \end{cases}$$

We shall solve this eigenvalue problem by considering cases.

- **Case 1:** Suppose $\lambda < 0$. Then, let $\lambda = -\mu^2$, where $\mu > 0$. So, we have the ODE

  $$X'' = \mu^2 X$$

  which yields the general solution

  $$X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

  Applying the Neumann boundary conditions $X'(0) = 0$ and $X'(l) = 0$, we have $C_1 = C_2$ and $C_1 \left(e^{\mu l} - e^{-\mu l}\right) = 0$ respectively. So, $C_1 = C_2 = 0$. Thus, only the trivial solution exists.

- **Case 2:** Suppose $\lambda = 0$. Then, it is easy to deduce that we have the constant eigenfunction $X(x) = 1$.

- **Case 3:** Suppose $\lambda > 0$. Let $\lambda = \beta^2$, where $\beta > 0$. Then,

  $$X'' + \beta^2 X = 0.$$

  It is easy to deduce that $C_2\beta = 0$ and $C_1\beta \sin(\beta l) = 0$. For a non-trivial solution, $C_1 \neq 0$ so $\sin(\beta l) = 0$. Thus, the positive eigenvalues are

  $$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{where } n \in \mathbb{N}$$

with eigenfunctions

$$X_n(x) = \cos\left(\frac{n\pi x}{l}\right).$$

Including the zero mode, the Neumann eigenfunctions are $X_0(x) = 1$ and $X_n(x) = \cos\left(\frac{n\pi x}{l}\right)$ where $n \in \mathbb{N}$.

For the Neumann heat problem, the time equation is

$$T'(t) = -k\lambda T(t).$$

For $\lambda_0 = 0$, we get $T_0'(t) = 0$ so $T_0(t) = A_0$. For

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{where } n \geq 1,$$

we get

$$T_n(t) = A_n e^{-k(n\pi/l)^2 t}.$$

Therefore the formal solution has the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k(n\pi/l)^2 t} \cos\left(\frac{n\pi x}{l}\right).$$

If the initial condition is $u(x,0) = \phi(x)$, then

$$\phi(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right).$$

Thus A_0, A_n are the Fourier cosine coefficients:

$$A_0 = \frac{1}{l}\int_0^l \phi(x)\, dx,$$

and for $n \geq 1$,

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)\, dx.$$

Theorem 4.1 (solution of Neumann heat problem). The formal solution of the Neumann heat problem (4.3) is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k(n\pi/l)^2 t} \cos\left(\frac{n\pi x}{l}\right)$$

The zero mode

$$A_0 = \frac{1}{l}\int_0^l \phi(x)\, dx$$

does not decay in time. It represents the average temperature. All other modes decay exponentially:

$$A_n e^{-k(n\pi/l)^2 t} \cos\left(\frac{n\pi x}{l}\right) \quad \text{where } n \geq 1.$$

Therefore, as $t \to \infty$, $u(x,t) \to A_0$. This matches the physical interpretation of insulated endpoints: heat cannot leave the interval, so the total heat is conserved, and the temperature eventually becomes uniform.

4.7 Complex Diffusion Coefficients and Schrödinger's Equation

The separation of variables method also works if the coefficient k is complex. For example, consider

$$u_t = iu_{xx}$$

This is a version of Schrödinger's equation. Here, $i = \sqrt{-1}$. With zero Neumann boundary conditions on $(0,l)$, the spatial eigenfunctions are still

$$X_n(x) = \cos\left(\frac{n\pi x}{l}\right) \quad \text{where } n = 0, 1, 2, \ldots.$$

The time equation becomes $T' = -i\lambda_n T$. Therefore,

$$T_n(t) = C_n e^{-i\lambda_n t}.$$

Since $|e^{-i\lambda_n t}| = 1$, the modes do not decay in magnitude. Instead, they oscillate in phase. Thus the separated solutions are

$$u_n(x,t) = C_n e^{-i\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi x}{l}\right).$$

Chapter 5

Fourier Series

5.1 Fourier Sine Series

Definition 5.1 (Fourier sine series). A series of the form

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$$

on the interval $[0,l]$ is called a Fourier sine series.

Suppose we want to represent a function $\phi(x)$ on $(0,l)$ as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

The goal is to find the coefficients A_n.

For functions $f,g:[0,l]\to\mathbb{R}$, define the inner product

$$(f,g) = \int_0^l f(x)g(x)\,dx.$$

We say that f and g are orthogonal if $(f,g)=0$.

Theorem 5.1 (Orthogonality of Fourier sine functions). For $m,n\in\mathbb{N}$,

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx = \begin{cases} \frac{l}{2} & \text{if } m=n; \\ 0, & \text{if } m\neq n. \end{cases}$$

Proof. Use the identity □

Proof. Using the product-to-sum identity

$$\sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b).$$

□

Now, suppose

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

Take the inner product with

$$\sin\left(\frac{m\pi x}{l}\right).$$

Then

$$\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)\,dx = \int_0^l \left[\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\right] \sin\left(\frac{m\pi x}{l}\right)\,dx.$$

Assuming the series converges uniformly, we can interchange integration and summation to obtain

$$\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)\,dx = \sum_{n=1}^{\infty} A_n \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)\,dx.$$

By orthogonality, all terms vanish except the term $n = m$. Hence

$$\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)\,dx = A_m \cdot \frac{l}{2} \quad \text{so} \quad A_m = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)\,dx.$$

Theorem 5.2 (Fourier sine coefficient formula). If

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \text{ on } (0,l) \quad \text{then} \quad A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

5.2 Heat and Wave Equations with Dirichlet Boundary Conditions

Consider the heat equation with Dirichlet boundary conditions (4.2). Using separation of variables, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right).$$

At $t = 0$,

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

Therefore, the coefficients are the Fourier sine coefficients of ϕ:

$$A_n = \frac{2}{l}\int_0^l \phi(y)\sin\left(\frac{n\pi y}{l}\right)\,dy.$$

Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l \phi(y)\sin\left(\frac{n\pi y}{l}\right)\,dy\right] e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi x}{l}\right).$$

Equivalently,

$$u(x,t) = \int_0^l H(x,y,t)\phi(y)\,dy,$$

where
$$H(x,y,t)=\sum_{n=1}^{\infty}\frac{2}{l}e^{-k\left(\frac{n\pi}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi y}{l}\right)$$
is the heat kernel for the interval $[0,l]$ with Dirichlet boundary conditions.

Next, consider the wave equation with Dirichlet boundary conditions (4.1). By separation of variables, the solution has the form
$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos(nt)+B_n\sin(nt))\sin(nx).$$
Using $u(x,0)=0$, we get $A_n=0$ for all $n\in\mathbb{N}$. Now differentiate with respect to t to obtain
$$u_t(x,t)=\sum_{n=1}^{\infty}(-nA_n\sin(nt)+nB_n\cos(nt))\sin(nx).$$
At $t=0$, since $A_n=0$, we get
$$u_t(x,0)=\sum_{n=1}^{\infty}nB_n\sin(nx).$$
But $u_t(x,0)=1$, so
$$1=\sum_{n=1}^{\infty}nB_n\sin(nx).$$
From the Fourier sine series of 1 on $(0,\pi)$, we know
$$nB_n=\frac{2}{\pi n}\left((-1)^{n+1}+1\right).$$
Therefore,
$$B_n=\frac{2}{\pi n^2}\left((-1)^{n+1}+1\right).$$
Thus, the solution is
$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^{n+1}+1\right)\sin(nt)\sin(nx).$$
Equivalently, since only odd terms survive,
$$u(x,t)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\sin((2k+1)t)\sin((2k+1)x).$$

5.3 Full Fourier Series on $(-l,l)$

We now consider the full Fourier series on the symmetric interval $(-l,l)$.

Definition 5.2 (full Fourier series). The full Fourier series of a function on $(-l,l)$ is a series of the form
$$\frac{A_0}{2}+\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi x}{l}\right)+B_n\sin\left(\frac{n\pi x}{l}\right)\right).$$

As usual, define the inner product on $(-l,l)$ by
$$(f,g)=\int_{-l}^{l}f(x)g(x)\,dx.$$

Theorem 5.3 (orthogonality of the full Fourier basis). The basis functions $1, \cos\left(\frac{n\pi x}{l}\right), \sin\left(\frac{n\pi x}{l}\right)$ for $n \in \mathbb{N}$ are mutually orthogonal on $(-l, l)$. More precisely, for $m, n \in \mathbb{N}$,

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) \, dx = 0.$$

Also,

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) \, dx = 0 \quad \text{if } m \neq n,$$

and

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) \, dx = 0 \quad \text{if } m \neq n.$$

For the norms, we have

$$\int_{-l}^{l} 1^2 \, dx = 2l,$$

and for $n \geq 1$,

$$\int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right) \, dx = l \quad \text{and} \quad \int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right) \, dx = l.$$

Suppose

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right).$$

Taking inner product with 1, we get

$$\int_{-l}^{l} \phi(x) \, dx = \frac{A_0}{2} \int_{-l}^{l} 1 \, dx.$$

Since

$$\int_{-l}^{l} 1 \, dx = 2l \quad \text{then} \quad A_0 = \frac{1}{l} \int_{-l}^{l} \phi(x) \, dx.$$

For $m \geq 1$, taking inner product with $\cos\left(\frac{m\pi x}{l}\right)$, we obtain a formula for A_m; similarly, taking inner product with $\sin\left(\frac{m\pi x}{l}\right)$, we obtain a formula for B_m. We summarise these in Theorem 5.4.

Theorem 5.4 (full Fourier coefficient formulas). If

$$\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right) \quad \text{on } (-l, l).$$

Then,

$$A_0 = \frac{1}{l} \int_{-l}^{l} \phi(x) \, dx,$$

and for $n \geq 1$,

$$A_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \cos\left(\frac{n\pi x}{l}\right) \, dx \quad \text{and} \quad B_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \sin\left(\frac{n\pi x}{l}\right) \, dx.$$

5.4 Fourier Cosine Series

Recall that for the heat equation with Neumann boundary condition (4.3), separation of variables gives solutions of the form

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi x}{l}\right).$$

If the initial condition is $u(x,0) = \phi(x)$, then we require then we require

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right).$$

This motivates the study of Fourier cosine series (Definition 5.3).

Definition 5.3 (Fourier cosine series). A series of the form

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)$$

on the interval $[0,l]$ is called a Fourier cosine series.

For Neumann boundary conditions, the relevant eigenvalue problem is

$$\begin{cases} X'' = -\lambda X & \text{if } 0 < x < l; \\ X'(0) = X'(l) = 0. \end{cases}$$

The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \text{ and } X_n(x) = \cos\left(\frac{n\pi x}{l}\right) \quad \text{where } n = 0, 1, 2, \ldots.$$

Here, $X_0(x) = 1$. The cosine functions satisfy the orthogonality relations

$$\int_0^l \cos\left(\frac{n\pi x}{l}\right) \cos\left(\frac{m\pi x}{l}\right) dx = 0 \quad \text{if } m \neq n.$$

Moreover,

$$\int_0^l 1^2 \, dx = l,$$

and for $n \geq 1$,

$$\int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx = \frac{l}{2}.$$

We then compute the Fourier cosine coefficients. Suppose

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right).$$

We want to find the coefficients A_n. First, take the inner product with 1. Then

$$\int_0^l \phi(x)\, dx = \int_0^l \left[\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\right] dx.$$

Since

$$\int_0^l \cos\left(\frac{n\pi x}{l}\right)\,dx = 0 \quad \text{for } n \geq 1,$$

we obtain

$$\int_0^l \phi(x)\,dx = \frac{1}{2}A_0 l.$$

Hence

$$A_0 = \frac{2}{l}\int_0^l \phi(x)\,dx.$$

Next, for $m \geq 1$, take the inner product with $\cos\left(\frac{m\pi x}{l}\right)$ so

$$\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right)\,dx = A_m \int_0^l \cos^2\left(\frac{m\pi x}{l}\right)\,dx.$$

Using

$$\int_0^l \cos^2\left(\frac{m\pi x}{l}\right)\,dx = \frac{l}{2},$$

we get

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{m\pi x}{l}\right)\,dx.$$

Theorem 5.5 (Fourier cosine coefficient formula). If

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{on } (0,l),$$

then

$$A_0 = \frac{2}{l}\int_0^l \phi(x)\,dx,$$

and for $n \geq 1$,

$$A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)\,dx.$$

5.5 Comparison between the Series

The Fourier sine series of ϕ on $(0,l)$ is

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad A_n = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi x}{l}\right)\,dx.$$

The sine functions are eigenfunctions of the Dirichlet eigenvalue problem

$$\begin{cases} X'' = -\lambda X & \text{if } 0 < x < l; \\ X(0) = X(l) = 0. \end{cases}$$

Thus Fourier sine series are naturally used for PDEs with zero Dirichlet boundary conditions.

On the other hand, the Fourier cosine series of ϕ on $(0,l)$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{where} \quad A_n = \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)\,dx.$$

Here $n = 0, 1, 2, \ldots$. However, the series is written with $\frac{1}{2}A_0$, so that the formula for A_0 remains consistent:

$$A_0 = \frac{2}{l}\int_0^l \phi(x)\,dx.$$

The cosine functions are eigenfunctions of the Neumann eigenvalue problem

$$\begin{cases} X'' = -\lambda X & \text{if } 0 < x < l; \\ X'(0) = X'(l) = 0. \end{cases}$$

Thus Fourier cosine series are naturally used for PDEs with zero Neumann boundary conditions.

We then discuss odd, even, and periodic extensions. Fourier sine and cosine series do not only describe the behaviour of a function on $(0, l)$. They also determine a natural extension of the function to the whole real line.

> **Definition 5.4 (periodic function).** A function f is periodic with period $T > 0$ if $f(x+T) = f(x)$ for all x.

For both Fourier sine and Fourier cosine series on $(0, l)$, the resulting extension has period $2l$. Indeed,

$$\sin\left(\frac{n\pi(x+2l)}{l}\right) = \sin\left(\frac{n\pi x}{l} + 2n\pi\right) = \sin\left(\frac{n\pi x}{l}\right) \quad \text{and} \quad \cos\left(\frac{n\pi(x+2l)}{l}\right) = \cos\left(\frac{n\pi x}{l} + 2n\pi\right) = \cos\left(\right.$$

Given a function ϕ on $(0, l)$, its odd extension to $(-l, l)$ is defined by

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l; \\ -\phi(-x) & \text{if } -l < x < 0. \end{cases}$$

The Fourier sine series gives the $2l$-periodic odd extension of ϕ. This is because sine is odd.

Next, given a function ϕ on $(0, l)$, its even extension to $(-l, l)$ is defined by

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & 0 < x < l; \\ \phi(-x) & \text{if } -l < x < 0. \end{cases}$$

The Fourier cosine series gives the $2l$-periodic even extension of ϕ. This is because cosine is even.

5.6 Complex Form of the Full Fourier Series

The real Fourier basis

$$\left\{1, \cos\left(\frac{n\pi x}{l}\right), \sin\left(\frac{n\pi x}{l}\right)\right\}$$

can be rewritten using complex exponentials. By Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, it is easy to see that the full Fourier series can be written using the complex basis $\left\{e^{\frac{in\pi x}{l}} : n \in \mathbb{Z}\right\}$.

Definition 5.5 (full Fourier series, complex form). The complex Fourier series of a function ϕ on $(-l,l)$ is

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}.$$

For complex-valued functions f, g, define

$$(f,g) = \int_{-l}^{l} f(x)\overline{g(x)}\,dx.$$

The conjugate is important. Without it, the inner product would not have the usual positivity property. For the complex basis functions $e_n(x) = e^{\frac{in\pi x}{l}}$, we have

$$(e_n,e_m) = \int_{-l}^{l} e^{\frac{in\pi x}{l}}\overline{e^{\frac{im\pi x}{l}}}\,dx = \int_{-l}^{l} e^{\frac{i(n-m)\pi x}{l}}\,dx.$$

If $n \neq m$, then the integral is equal to 0; otherwise the integral is equal to $2l$.

Suppose

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}.$$

Taking the complex inner product with $e_m(x) = e^{\frac{im\pi x}{l}}$, we obtain

$$(\phi,e_m) = \sum_{n=-\infty}^{\infty} c_n(e_n,e_m).$$

By orthogonality, all terms vanish except $n = m$, so

$$(\phi,e_m) = 2lc_m.$$

Therefore,

$$c_m = \frac{1}{2l}\int_{-l}^{l} \phi(x)e^{-\frac{im\pi x}{l}}\,dx.$$

Theorem 5.6 (Complex Fourier coefficient formula). If

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \quad \text{then} \quad c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x)e^{-\frac{in\pi x}{l}}\,dx.$$

5.7 Orthogonality of Eigenfunctions

The orthogonality of sine, cosine, and complex exponential functions is not a coincidence. These functions are eigenfunctions of the differential equation

$$X'' = -\lambda X$$

with different boundary conditions. The key principle is that eigenfunctions corresponding to different eigenvalues are orthogonal, provided the boundary conditions are symmetric.

Consider the eigenvalue problem

$$X'' = -\lambda X \quad \text{on } (a,b).$$

We say the boundary condition is symmetric if any two eigenfunctions f, g satisfy

$$\left[f'(x)g(x) - f(x)g'(x)\right]_a^b = 0.$$

Typical examples include:

- Dirichlet boundary condition: $X(a) = X(b) = 0$
- Neumann boundary condition: $X'(a) = X'(b) = 0$
- Periodic boundary condition: $X(a) = X(b)$ and $X'(a) = X'(b)$
- Mixed Robin-type boundary conditions under suitable symmetry assumptions

Theorem 5.7 (orthogonality under symmetric boundary conditions). Consider

$$X'' = -\lambda X \quad \text{on } (a,b),$$

with symmetric boundary conditions. If f and g are real eigenfunctions with distinct eigenvalues $\lambda_1 \neq \lambda_2$, then

$$\int_a^b f(x)g(x)\,dx = 0.$$

Proof. Suppose

$$f'' = -\lambda_1 f \quad \text{and} \quad g'' = -\lambda_2 g.$$

Then,

$$\int_a^b f''g\,dx = -\lambda_1 \int_a^b fg\,dx \quad \text{and} \quad \int_a^b fg''\,dx = -\lambda_2 \int_a^b fg\,dx.$$

So,

$$\int_a^b (f''g - fg'')\,dx = -(\lambda_1 - \lambda_2) \int_a^b fg\,dx.$$

Since

$$f''g - fg'' = \frac{d}{dx}(f'g - fg'),$$

then

$$\int_a^b (f''g - fg'')\,dx = \left[f'g - fg'\right]_a^b.$$

By the symmetric boundary condition, $[f'g - fg']_a^b = 0$ which implies

$$-(\lambda_1 - \lambda_2) \int_a^b fg\,dx = 0.$$

Since $\lambda_1 \neq \lambda_2$, we conclude that

$$\int_a^b fg\,dx = 0.$$

Hence f and g are orthogonal. □

5.8 Robin Boundary Conditions and Sturm-Liouville Theory

A useful class of boundary conditions is given by Robin-type boundary conditions. Consider

$$\begin{cases} X''(x) = -\lambda X(x) & \text{if } 0 < x < \pi; \\ -c_1 X'(0) + d_1 X(0) = 0 \\ c_2 X'(\pi) + d_2 X(\pi) = 0 \end{cases} \tag{5.1}$$

where $c_1, c_2, d_1, d_2 \geq 0$ and $c_i + d_i > 0$ for $i = 1, 2$.

Theorem 5.8 (Sturm-Liouville type properties). For the eigenvalue problem (5.1), the following hold:

- **(i)** All eigenvalues are non-negative real numbers
- **(ii)** If $d_1 + d_2 > 0$, then all eigenvalues are > 0
- **(iii)** Eigenfunctions corresponding to distinct eigenvalues are orthogonal
- **(iv)** The eigenvalues form an increasing sequence

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = +\infty$$

This theorem is a special case of the Sturm-Liouville theorem. Its main message is that the good orthogonality properties of sine and cosine functions come from the structure of the differential operator and boundary conditions, not from the explicit formulas alone.

For complex-valued eigenfunctions, we use the complex inner product

$$(f, g) = \int_a^b f(x)\overline{g(x)}\, dx.$$

Suppose

$$X'' = -\lambda X.$$

Multiplying by $\overline{X}$ and integrating gives

$$\int_a^b X''\overline{X}\, dx = -\lambda \int_a^b |X|^2\, dx.$$

Using integration by parts,

$$\int_a^b X''\overline{X}\, dx = \left[X'\overline{X}\right]_a^b - \int_a^b |X'|^2\, dx.$$

Thus,

$$\lambda \int_a^b |X|^2\, dx = \int_a^b |X'|^2\, dx - \left[X'\overline{X}\right]_a^b.$$

Under symmetric boundary conditions, the boundary term is real, and in many standard cases it vanishes. Hence λ is real. Under suitable positive boundary conditions, one further obtains $\lambda \geq 0$.

The previous results fit into a broader framework involving self-adjoint operators. Let L be a differential operator on a domain D, together with some boundary condition. The eigenvalue problem is

$$LX = \lambda X.$$

Definition 5.6 (self-adjoint operator). Let L be a differential operator with a given boundary condition. We say that L is self-adjoint if for any two functions f, g satisfying the boundary condition,

$$(Lf, g) = (f, Lg).$$

For complex-valued functions, the inner product is understood as

$$(f, g) = \int_D f\overline{g}.$$

Theorem 5.9 (spectral properties of self-adjoint operators). Suppose L is self-adjoint. Then, the following hold:

- **(i)** All eigenvalues of L are real
- **(ii)** Eigenfunctions corresponding to distinct eigenvalues are orthogonal
- **(iii)** If $(Lf, f) \geq 0$ for all admissible f, then all eigenvalues are ≥ 0

Proof. Suppose $Lf = \lambda f$. Then, $(Lf, f) = (\lambda f, f) = \lambda(f, f)$. Since L is self-adjoint, then $(Lf, f) = (f, Lf)$. However, we have

$$(f, Lf) = (f, \lambda f) = \overline{\lambda}(f, f) \quad \text{which implies} \quad \lambda(f, f) = \overline{\lambda}(f, f).$$

Since $f \neq 0$, then $(f, f) > 0$. So, $\lambda = \overline{\lambda}$, which implies that all eigenvalues of L are real, proving **(i)**.

We then prove **(ii)**. Suppose

$$Lf = \lambda_1 f \quad \text{and} \quad Lg = \lambda_2 g.$$

Then $(Lf, g) = \lambda_1(f, g)$ while by self-adjointness, $(Lf, g) = (f, Lg) = \lambda_2(f, g)$ since $\lambda_2 \in \mathbb{R}$. Hence, $(\lambda_1 - \lambda_2)(f, g) = 0$. If $\lambda_1 \neq \lambda_2$, then $(f, g) = 0$. Thus eigenfunctions with distinct eigenvalues are orthogonal. This proves **(ii)**.

Finally we prove **(iii)**. If $(Lf, f) \geq 0$ and $Lf = \lambda f$, then $(Lf, f) = \lambda(f, f) \geq 0$. Since $(f, f) > 0$, we obtain $\lambda \geq 0$. □

Example 5.1 (the Laplacian on a drumhead). Consider the eigenvalue problem

$$\begin{cases} -\Delta U(x,y) = \lambda U(x,y) & \text{if } (x,y) \in D; \\ U(x,y) = 0 & \text{if } (x,y) \in \partial D. \end{cases} \tag{5.2}$$

This appears when studying the vibration of a drumhead. The corresponding wave equation is

$$u_{tt} = \Delta u.$$

Using separation of variables,

$$u(x,y,t) = T(t)U(x,y),$$

we obtain

$$T''(t)U(x,y) = T(t)\Delta U(x,y).$$

If $\Delta U = -\lambda U$, then $T'' = -\lambda T$. Hence,

$$T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t).$$

Therefore, each eigenvalue λ_n produces a vibrating mode

$$u_n(x,y,t) = \left[A_n\cos(\sqrt{\lambda_n}t) + B_n\sin(\sqrt{\lambda_n}t)\right]U_n(x,y).$$

When one hears the sound of a drum, one is essentially hearing the frequencies $\sqrt{\lambda_n}$, where λ_n are the eigenvalues of the drumhead.

We now discuss the self-adjointness of $-\Delta$. Let $L = -\Delta$. For functions u, v satisfying zero Dirichlet boundary condition, $u = v = 0$ on ∂D. Green's identity gives

$$(-\Delta u, v) = \int_D \nabla u \cdot \overline{\nabla v}\, dx = (u, -\Delta v).$$

Thus $-\Delta$ with zero Dirichlet boundary condition is self-adjoint. Moreover,

$$(-\Delta u, u) = \int_D |\nabla u|^2\, dx \geq 0.$$

Hence all eigenvalues are non-negative. In fact, for the zero Dirichlet problem, the eigenvalues are strictly positive. If $\lambda = 0$, then

$$\int_D |\nabla u|^2\, dx = 0.$$

Thus, $\nabla u = 0$ so u is constant. Since $u = 0$ on ∂D, it follows that $u \equiv 0$ contradicting the requirement that an eigenfunction is non-zero. Therefore, $\lambda > 0$.

For the mentioned Dirichlet problem (5.2), there is an infinite family of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = +\infty.$$

Although explicit formulas are generally unavailable for arbitrary domains $D \subseteq \mathbb{R}^n$, Weyl's law describes the asymptotic distribution of eigenvalues:

$$\lim_{A\to\infty} \frac{\#\{\text{eigenvalues less than } A\}}{A^{n/2}} = (2\pi)^{-n}\omega_n \operatorname{vol}(D),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

A natural question is:

> Is the shape of a domain $D \subseteq \mathbb{R}^2$ uniquely determined, up to translation and rotation, by its eigenvalues
>
> $$\lambda_1 \leq \lambda_2 \leq \cdots ?$$

Equivalently, can one hear the shape of a drum? This question was famously posed in 1966. Counterexamples were constructed for polygonal domains in 1992, showing that different polygonal drums can have the same eigenvalues. However, the question remains more subtle for domains with smooth boundary.

5.9 Convergence of Fourier Series

So far, we have formally written functions as Fourier series. For example, given a function ϕ on an interval (a,b), we may formally write

$$\phi(x) = \sum_n A_n X_n(x),$$

where X_n are eigenfunctions corresponding to some symmetric boundary condition, such as Dirichlet, Neumann, or periodic boundary conditions. A natural question is:

> Does the Fourier series actually converge to the original function ϕ?

To answer this properly, we need to distinguish between different notions of convergence.

> **Definition 5.7 (pointwise convergence).** We say that an infinite series
>
> $$\sum_{n=1}^{\infty} f_n(x)$$
>
> converges pointwise to $\phi(x)$ on (a,b) if
>
> $$\lim_{N\to\infty}\left(\sum_{n=1}^{N} f_n(x) - \phi(x)\right) = 0 \quad \text{for } x \in (a,b).$$

Pointwise convergence means that for each fixed point x, the sequence of partial sums

$$S_N(x) = \sum_{n=1}^{N} f_n(x)$$

converges to $\phi(x)$. However, the speed of convergence may depend on x. Hence pointwise convergence is relatively weak.

Definition 5.8 (Uniform convergence). We say that an infinite series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly to $\phi(x)$ on $[a,b]$ if

$$\lim_{N\to\infty} \max_{x\in[a,b]} \left| \sum_{n=1}^{N} f_n(x) - \phi(x) \right| = 0.$$

Uniform convergence means that the partial sums approach ϕ uniformly over the whole interval. In other words, after N is sufficiently large, the whole graph of S_N is close to the graph of ϕ.

Pointwise convergence allows the convergence speed to depend on x. Uniform convergence requires one N that works for all x simultaneously.

Example 5.2 (pointwise but not uniform convergence). Consider

$$f_n(x) = x^n - x^{n-1} \quad \text{where } n \in \mathbb{N} \text{ and } x \in [0,1].$$

Then the N^{th} partial sum is

$$S_N(x) = \sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} \left(x^n - x^{n-1}\right) = x^N - 1.$$

For $0 \le x < 1$,

$$\lim_{N\to\infty} x^N = 0 \quad \text{so} \quad \lim_{N\to\infty} S_N(x) = -1.$$

At $x = 1$, $S_N(1) = 0$, so the pointwise limit is

$$g(x) = \begin{cases} -1 & 0 \le x < 1; \\ 0 & x = 1. \end{cases}$$

Thus, $S_N(x) \to g(x)$ pointwise on $[0,1]$. However, the convergence is not uniform. Indeed, for $0 \le x < 1$,

$$|S_N(x) - g(x)| = |x^N - 1 - (-1)| = x^N \quad \text{so} \quad \max_{x\in[0,1)} x^N = 1.$$

So, the maximum error does not tend to 0. Therefore, $S_N \not\to g$ uniformly on $[0,1]$.

5.10 Uniform Convergence of Fourier Series

Let $\phi(x)$ be a function on $[a,b]$. Suppose we construct a Fourier series using a symmetric boundary condition, such as Dirichlet, Neumann, or periodic boundary conditions.

Theorem 5.10 (uniform convergence of Fourier series). Suppose $\phi(x)$ satisfies the given boundary conditions, and $\phi \in \mathcal{C}^2([a,b])$. That is, ϕ, ϕ', and ϕ'' exist and are continuous on $[a,b]$. Then the corresponding Fourier series

$$\sum_n A_n X_n(x)$$

converges uniformly to $\phi(x)$ on $[a,b]$.

The phrase 'given boundary conditions' refers to the boundary conditions used to generate the Fourier basis.

- For a Fourier sine series, the corresponding boundary condition is zero Dirichlet: $\phi(a) = \phi(b) = 0$
- For a Fourier cosine series, the corresponding boundary condition is zero Neumann: $\phi'(a) = \phi'(b) = 0$
- For a full Fourier series, the corresponding boundary condition is periodic: $\phi(a) = \phi(b)$ and $\phi'(a) = \phi'(b)$

Example 5.3. Let

$$\phi(x) = x(1-x) \quad \text{where } 0 \leq x \leq 1.$$

Consider its Fourier sine series on $[0,1]$. Since $\phi(0) = 0$ and $\phi(1) = 0$, then ϕ satisfies the zero Dirichlet boundary condition. Also, $\phi(x) = x(1-x)$ is a polynomial, so $\phi \in \mathcal{C}^2([0,1])$. Therefore, by the uniform convergence theorem, the Fourier sine series of ϕ converges uniformly to ϕ on $[0,1]$.

5.11 Pointwise Convergence of Fourier Series

Theorem 5.11 (pointwise convergence of Fourier series). Suppose f and f' are both piecewise continuous on $[a,b]$. Then the Fourier series of f converges pointwise at every $x \in \mathbb{R}$. The sum is

$$\sum_n A_n X_n(x) = \frac{1}{2}\left[f_{\text{ext}}(x^+) + f_{\text{ext}}(x^-)\right],$$

where f_{ext} is the suitable extension of f to $\mathbb{R}$. This extension may be periodic, odd periodic, or even periodic, depending on which Fourier series is used.

Thus,

- If f_{ext} is continuous at x, then the Fourier series converges to $f_{\text{ext}}(x)$
- If f_{ext} has a jump discontinuity at x, then the Fourier series converges to the average of the left and right limits:

$$\frac{1}{2}\left[f_{\text{ext}}(x^+) + f_{\text{ext}}(x^-)\right]$$

Example 5.4 (full Fourier series of e^x on $[-1,1)$.]Letf(x)=e^x, where $-1<x<1$. The full Fourier series corresponds to the periodic extension with period 2. That is, $f_{\text{ext}}(x+2)=f_{\text{ext}}(x)$. At interior points $-1<x<1$, the function is continuous, so the Fourier series converges to e^x.

At the endpoints $x=1+2n$ and $x=-1+2n$, there is a jump discontinuity. The left and right limits are e and e^{-1}. Therefore the Fourier series converges to $\frac{1}{2}(e+e^{-1})$ at those jump points.

5.12 L^2 Convergence of Fourier Series

Definition 5.9 (L^2 norm). For a function f on $[a,b]$, its L^2 norm is

$$\|f\|_{L^2}=\left(\int_a^b |f(x)|^2\,dx\right)^{1/2}.$$

The L^2 distance between f and g is

$$\|f-g\|_{L^2}=\left(\int_a^b |f(x)-g(x)|^2\,dx\right)^{1/2}.$$

This measures the mean-square error between two functions.

Theorem 5.12 (L^2 convergence of Fourier series). For any L^2 function $\phi(x)$ on (a,b), its Fourier series converges to ϕ in the L^2 sense. That is, if

$$S_N(x)=\sum_{n=1}^{N} A_nX_n(x) \quad \text{then} \quad \lim_{N\to\infty}\|S_N-\phi\|_{L^2}=0.$$

Note that uniform convergence is stronger than L^2 convergence. L^2 convergence does not require pointwise convergence at every point.

Now, let $\{X_n\}$ be an orthogonal family of Fourier basis functions. Suppose

$$\phi(x)=\sum_{n=1}^{\infty} A_nX_n(x).$$

Using orthogonality, $(X_n,X_m)=0$ for $n\neq m$. Taking L^2 norms gives the infinite-dimensional analogue of the Pythagorean theorem, known as Parseval's identity (Theorem 5.13).

Theorem 5.13 (Parseval's identity). If $\phi\in L^2([a,b])$, then

$$\sum_{n=1}^{\infty}|A_n|^2\int_a^b |X_n(x)|^2\,dx=\int_a^b |\phi(x)|^2\,dx.$$

5.13 The Fourier Transform

Fourier series represent functions on bounded intervals or periodic domains. The Fourier transform arises when we let the length of the interval tend to infinity. Recall from Definition 5.5 that for a function f on $(-l,l)$, its complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \quad \text{where} \quad c_n = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-\frac{in\pi y}{l}}\,dy.$$

Let $k = \frac{n\pi}{l}$. As $l \to \infty$, the discrete frequencies become continuous. This motivates the Fourier transform (Definition 5.10).

Definition 5.10 (Fourier transform). The Fourier transform of f is

$$\widehat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \quad \text{where } k \in \mathbb{R}.$$

The variable k is called the frequency variable.

Definition 5.11 (inverse Fourier transform). The function f can be recovered from $\widehat{f}$ by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)e^{ikx}\,dk.$$

The Fourier transform version of Parseval's identity is Plancherel's theorem (Theorem 5.14).

Theorem 5.14 (Plancherel's theorem). If $f \in L^2(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{f}(k)|^2\,dk.$$

Thus the Fourier transform preserves L^2 energy up to the factor 2π.

Definition 5.12 (convolution). For two functions f and g, their convolution is

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Proposition 5.1 (Fourier transform of convolution). The Fourier transform converts convolution into multiplication:

$$\widehat{f*g}(k) = \widehat{f}(k)\widehat{g}(k).$$

Proof. We compute:

$$\begin{aligned}\widehat{f*g}(k) &= \int_{-\infty}^{\infty} (f*g)(x)e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right] e^{-ikx}\,dx.\end{aligned}$$

Let

$$z = x - y.$$

Then $x = z + y$, and

$$e^{-ikx} = e^{-ik(z+y)} = e^{-ikz}e^{-iky}.$$

Thus

$$\begin{aligned}
\widehat{f*g}(k) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)g(y)e^{-ikz}e^{-iky}\,dzdy \\
&= \left(\int_{-\infty}^{\infty} f(z)e^{-ikz}\,dz\right)\left(\int_{-\infty}^{\infty} g(y)e^{-iky}\,dy\right) \\
&= \widehat{f}(k)\widehat{g}(k).
\end{aligned}$$

□

The Fourier transform is useful for solving PDEs on the whole line $\mathbb{R}$. The general procedure is as follows:

- **Step 1:** Start with the original PDE.
- **Step 2:** Take the Fourier transform in the spatial variable x.
- **Step 3:** The PDE is transformed into an ODE in time t with frequency parameter k.
- **Step 4:** Solve the transformed ODE.
- **Step 5:** Apply the inverse Fourier transform to recover $u(x,t)$.

Consider the heat equation on the whole line:

$$\begin{cases} S_t = cS_{xx} & \text{where } x \in \mathbb{R} \text{ and } t > 0; \\ S(x,0) = \delta(x). \end{cases}$$

Here $c > 0$. We want to derive the fundamental solution of the heat equation. Take the Fourier transform in x:

$$\widehat{S}(k,t) = \int_{-\infty}^{\infty} S(x,t)e^{-ikx}\,dx.$$

Using $\widehat{S_{xx}}(k,t) = -k^2\widehat{S}(k,t)$, the PDE becomes

$$\frac{\partial}{\partial t}\widehat{S}(k,t) = -ck^2\widehat{S}(k,t).$$

For each fixed k, this is an ODE in t: $\widehat{S}_t = -ck^2\widehat{S}$. Solving, we obtain

$$\widehat{S}(k,t) = e^{-ck^2t}\widehat{S}(k,0).$$

Since $S(x,0) = \delta(x)$, we have $\widehat{S}(k,0) = 1$ so

$$\widehat{S}(k,t) = e^{-ck^2t}.$$

Using the inverse Fourier transform of a Gaussian,

$$\mathcal{F}^{-1}\left(e^{-bk^2}\right) = \frac{1}{\sqrt{4\pi b}}e^{-\frac{x^2}{4b}},$$

with $b = ct$, we obtain
$$S(x,t) = \frac{1}{\sqrt{4\pi ct}} e^{-\frac{x^2}{4ct}}.$$
Equivalently,
$$S(x,t) = \frac{1}{2\sqrt{\pi ct}} e^{-\frac{x^2}{4ct}}.$$

Theorem 5.15. The fundamental solution of the heat equation $u_t = cu_{xx}$ on $\mathbb{R}$ is
$$S(x,t) = \frac{1}{\sqrt{4\pi ct}} e^{-\frac{x^2}{4ct}}.$$

Now consider
$$\begin{cases} u_t = cu_{xx} & \text{if } x \in \mathbb{R} \text{ and } t > 0; \\ u(x,0) = \phi(x). \end{cases}$$
Taking Fourier transform in x, we get
$$\widehat{u}_t(k,t) = -ck^2 \widehat{u}(k,t) \quad \text{so} \quad \widehat{u}(k,t) = e^{-ck^2 t} \widehat{\phi}(k).$$
Since $\widehat{S}(k,t) = e^{-ck^2 t}$, we have
$$\widehat{u}(k,t) = \widehat{S}(k,t)\widehat{\phi}(k).$$
By Proposition 5.1, $u(x,t) = (S(\cdot,t) * \phi)(x)$. Hence,
$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy.$$
Therefore,
$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi ct}} e^{-\frac{(x-y)^2}{4ct}} \phi(y)\, dy.$$
For the heat equation,
$$\widehat{u}(k,t) = e^{-ck^2 t} \widehat{\phi}(k).$$
The factor $e^{-ck^2 t}$ decays in time. Moreover, high frequencies, meaning large $|k|$, decay faster because the exponent contains $-ck^2 t$. Thus the heat equation smooths the initial data: high-frequency oscillations are rapidly damped. By Plancherel's theorem (Theorem 5.14),
$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{u}(k,t)|^2\, dk.$$
Since $|\widehat{u}(k,t)|^2 = e^{-2ck^2 t} |\widehat{\phi}(k)|^2$, the L^2-energy decreases as t increases.

Next, we consider the wave equation
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{if } x \in \mathbb{R} \text{ and } t > 0; \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

The classical d'Alembert formula (Theorem 3.8) is

$$u(x,t) = \frac{1}{2}[\phi(x-ct)+\phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy.$$

Fourier transform gives another derivation. Taking Fourier transform in x, we obtain

$$\widehat{u}_{tt}(k,t) = -c^2k^2\widehat{u}(k,t).$$

Thus, $\widehat{u}_{tt} + c^2k^2\widehat{u} = 0$. For each fixed k, this is a harmonic oscillator equation in t. The general solution is

$$\widehat{u}(k,t) = A(k)\cos(ckt) + B(k)\sin(ckt).$$

Using $\widehat{u}(k,0) = \widehat{\phi}(k)$, we get $A(k) = \widehat{\phi}(k)$. Also,

$$\widehat{u}_t(k,t) = -ckA(k)\sin(ckt) + ckB(k)\cos(ckt).$$

At $t = 0$, $\widehat{u}_t(k,0) = ckB(k)$. Since $\widehat{u}_t(k,0) = \widehat{\psi}(k)$, we get

$$B(k) = \frac{\widehat{\psi}(k)}{ck}.$$

Therefore,

$$\widehat{u}(k,t) = \widehat{\phi}(k)\cos(ckt) + \frac{\widehat{\psi}(k)}{ck}\sin(ckt).$$

Applying the inverse Fourier transform recovers the d'Alembert solution.

Chapter 6

Harmonic Functions

6.1 The Laplace Equation

For a domain $D \subseteq \mathbb{R}^n$, the Laplace equation is $\Delta u = 0$ in D. Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

In dimension 2,

$$\Delta u = u_{xx} + u_{yy},$$

and in dimension 3,

$$\Delta u = u_{xx} + u_{yy} + u_{zz}.$$

> Definition 6.1 (harmonic function). A function $u : D \to \mathbb{R}$ is called **harmonic** in D if $\Delta u = 0$ in D.

The Laplace equation is the standard model of an elliptic equation. The Laplace equation appears naturally in many problems.

- **Stationary heat and wave equations:** If u is independent of time, then the heat equation $u_t = \Delta u$ reduces to $\Delta u = 0$. Thus stationary temperature distributions are harmonic functions. Similarly, a stationary solution of the wave equation $u_{tt} = \Delta u$ also satisfies $\Delta u = 0$.
- **Steady incompressible irrotational fluid flow:** Let $\mathbf{v}$ be a velocity field. The flow is incompressible if $\operatorname{div} \mathbf{v} = 0$. If the flow is irrotational, then $\operatorname{curl} \mathbf{v} = \mathbf{0}$. In many cases, this implies that there is a scalar potential φ such that $\mathbf{v} = \nabla\varphi$. Then incompressibility gives

  $$0 = \operatorname{div} \mathbf{v} = \operatorname{div}(\nabla\varphi) = \Delta\varphi.$$

  Therefore the velocity potential φ is harmonic.

- **Complex analysis:** If $f(z) = u(x,y) + iv(x,y)$ is holomorphic, then u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. Differentiating gives $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Since mixed partial derivatives agree by Clairaut's theorem, $u_{xx} + u_{yy} = 0$. So, $\Delta u = 0$ and $\Delta v = 0$. Hence the real and imaginary parts of holomorphic functions are harmonic.

- **Random walks and Brownian motion:** Let $D \subseteq \mathbb{R}^2$ be a domain and let $\Gamma \subseteq \partial D$. Suppose a random walker starts at $(x,y) \in D$ and moves to one of the neighbouring grid points with equal probability. Let $p(x,y)$ be the probability that the walker eventually exits D through Γ. The discrete averaging property gives

$$p(x,y) = \frac{1}{4}\left[p(x+\Delta x,y) + p(x-\Delta x,y) + p(x,y+\Delta x) + p(x,y-\Delta x)\right].$$

Taylor expansion gives

$$p(x+\Delta x,y) + p(x-\Delta x,y) = 2p(x,y) + p_{xx}(x,y)(\Delta x)^2 + o((\Delta x)^2),$$

and similarly for the y-direction. Hence,

$$0 = \frac{1}{4}\left[p_{xx} + p_{yy}\right](\Delta x)^2 + o((\Delta x)^2).$$

Letting $\Delta x \to 0$, we obtain $\Delta p = 0$. Thus exit probabilities for Brownian motion are harmonic.

6.2 Calculus of Variations

Suppose we want to solve a PDE written abstractly as $A[u] = 0$, where A is a differential operator. In the calculus of variations, one tries to find an energy functional J such that $A[u] = J'[u]$. Then the PDE $A[u] = 0$ can be interpreted as the critical point equation $J'[u] = 0$. Thus solving the PDE is related to finding minimisers or critical points of an energy functional.

Definition 6.2 (functional). A functional is a map from a set of functions to real numbers. For example,

$$J(f) = \int_a^b f(x)dx$$

is a functional on $\mathcal{C}([a,b])$.

Definition 6.3 (variational problem). A variational problem asks for maxima or minima of a functional J over an admissible class of functions.

Definition 6.4 (Euler-Lagrange equation). The Euler-Lagrange equation is the PDE obtained by setting the first variation of a functional equal to zero.

Let u be a candidate minimizer of a functional J. Let η be a test function satisfying the homogeneous boundary condition, and consider the perturbation $u+\varepsilon\eta$. Define

$$j(\varepsilon) = J(u+\varepsilon\eta).$$

If u is a local minimiser, then $j(\varepsilon) \geq j(0)$ for all sufficiently small ε. Hence, $j'(0) = 0$. This derivative is called the first variation:

$$\delta J(u)(\eta) = \frac{d}{d\varepsilon} J(u+\varepsilon\eta)\bigg|_{\varepsilon=0}.$$

Thus a necessary condition for u to be a minimiser is $\delta J(u)(\eta) = 0$ for all admissible variations η.

We now discuss minimal surfaces and the Dirichlet functional. Let Γ be a closed simple curve in $\mathbb{R}^3$. Suppose a surface S is given as the graph of a function $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. The surface area is

$$A(S) = \int_D \sqrt{1+u_x^2+u_y^2}\, dxdy.$$

Thus the area functional is

$$J(u) = \int_D \sqrt{1+u_x^2+u_y^2}\, dxdy.$$

The minimal surface problem asks for u minimising J among all functions with prescribed boundary values. For small gradients, use the expansion

$$\sqrt{1+s} = 1 + \frac{1}{2}s + \cdots.$$

Then

$$\sqrt{1+u_x^2+u_y^2} \approx 1 + \frac{1}{2}(u_x^2+u_y^2).$$

Ignoring the constant term and higher-order terms leads to the Dirichlet functional

$$\widetilde{J}(u) = \frac{1}{2}\int_D \left(u_x^2+u_y^2\right)\, dxdy = \frac{1}{2}\int_D |\nabla u|^2\, dxdy.$$

Let

$$M_g = \{w \in C^2(\Omega) : w|_{\partial\Omega} = g\}.$$

Consider the Dirichlet functional

$$\widetilde{J}(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\, dx.$$

Suppose $u \in M_g$ minimises $\widetilde{J}$. Let $\eta \in C_c^\infty(\Omega)$ or more generally let η satisfy $\eta\,|_{\partial\Omega} = 0$. Then, $u+\varepsilon\eta \in M_g$. Define

$$j(\varepsilon) = \widetilde{J}(u+\varepsilon\eta).$$

Then,

$$j(\varepsilon) = \frac{1}{2}\int_\Omega |\nabla u + \varepsilon\nabla\eta|^2\, dx.$$

Expanding, we have

$$j(\varepsilon) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla \eta + \varepsilon^2 |\nabla \eta|^2\right)\, dx.$$

Therefore,

$$j'(0) = \int_\Omega \nabla u \cdot \nabla \eta \, dx.$$

Since u is a minimiser, then $j'(0) = 0$. Hence,

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx = 0$$

for every η vanishing on the boundary. Using integration by parts,

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx = -\int_\Omega \eta \Delta u \, dx + \int_{\partial\Omega} \eta \frac{\partial u}{\partial n}\, dS.$$

Since $\eta = 0$ on $\partial\Omega$, the boundary term vanishes. Hence

$$\int_\Omega \eta \Delta u \, dx = 0$$

for all test functions η. Therefore, $\Delta u = 0$ in Ω.

> The Euler-Lagrange equation of the Dirichlet functional
>
> $$\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx$$
>
> is the Laplace equation $\Delta u = 0$.

6.3 Dirichlet Principle and Poisson's Equation

Now let f be given and consider the functional

$$J(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - wf\right)\, dx$$

on

$$M_g = \{w \in C^2(\Omega) : w\,|_{\partial\Omega} = g\}.$$

Let u be a minimiser. For variations $u + \varepsilon\eta$ with $\eta\,|_{\partial\Omega} = 0$, define $j(\varepsilon) = J(u + \varepsilon\eta)$. Then,

$$j(\varepsilon) = \int_\Omega \left[\frac{1}{2}|\nabla u + \varepsilon \nabla \eta|^2 - (u + \varepsilon\eta) f\right]\, dx.$$

Differentiating at $\varepsilon = 0$,

$$j'(0) = \int_\Omega (\nabla u \cdot \nabla \eta - \eta f)\, dx.$$

Since u is a minimiser, $j'(0) = 0$ so

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx = \int_\eta \, dx.$$

Using integration by parts,

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx = -\int_\Omega \eta \Delta u \, dx,$$

because $\eta = 0$ on $\partial\Omega$. Hence,

$$-\int_\Omega \eta \Delta u \, dx = \int_\Omega f \eta \, dx.$$

Thus

$$\int_\Omega (-\Delta u - f)\eta \, dx = 0$$

for all admissible η. Therefore, $-\Delta u = f$ in Ω. Together with the boundary condition, we obtain the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Suppose u_1 and u_2 are two solutions to this Dirichlet problem. Let $w = u_1 - u_2$. Then,

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiply by w and integrate to obtain

$$0 = \int_\Omega (-\Delta w) w \, dx.$$

Integrating by parts,

$$\int_\Omega (-\Delta w) w \, dx = \int_\Omega |\nabla w|^2 \, dx - \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, dS.$$

Since $w = 0$ on $\partial\Omega$, the boundary term vanishes. Therefore,

$$\int_\Omega |\nabla w|^2 \, dx = 0.$$

Hence $\nabla w = 0$ in Ω. Thus w is constant. Since $w = 0$ on $\partial\Omega$, we get $w \equiv 0$ so $u_1 = u_2$.

Proposition 6.1. The Dirichlet problem for Poisson's equation has at most one solution.

6.4 Maximum Principle for Harmonic Functions

Theorem 6.1 (maximum principle). Let $D \subseteq \mathbb{R}^n$ be bounded. Suppose

$$u \in C^2(D) \cap C(\overline{D}) \quad \text{and} \quad \Delta u = 0 \text{ in } D.$$

Then,

$$\max_{\overline{D}} u = \max_{\partial D} u \quad \text{and} \quad \min_{\overline{D}} u = \min_{\partial D} u.$$

Moreover, unless u is constant, the maximum and minimum cannot be attained in the interior.

6.5 Laplace Operator in Polar and Spherical Coordinates

Proposition 6.2. The Laplace equation satisfies the following properties:

(i) Translation invariance: Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic, so $\Delta u = 0$. For a fixed vector $a \in \mathbb{R}^n$, define $v(x) = u(x+a)$. Then, $\Delta v(x) = \Delta u(x+a) = 0$ so v is also harmonic.

(ii) Rotational invariance: Let $\mathbf{B}$ be an orthogonal matrix. Define $v(x) = u(\mathbf{B}x)$. The gradient transforms as $\nabla v(x) = \mathbf{B}^T(\nabla u)(\mathbf{B}x)$. Taking divergence gives $\Delta v(x) = \Delta u(\mathbf{B}x)$. Therefore, if u is harmonic, then v is harmonic.

The Laplace equation is invariant under translations and rotations. This is why polar and spherical coordinates are especially useful for radially symmetric domains.

Note that polar coordinates in two dimensions is given by $x = r\cos\theta$ and $y = r\sin\theta$. Then the Laplacian in polar coordinates is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

As for spherical coordinates in three dimensions, we have $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. Then, the Laplacian is

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\varphi\varphi}\right].$$

We shall solve Laplace's equation on a rectangle. Let $D = (0,a)\times(0,b)$. Consider the subproblem

$$\begin{cases} \Delta u = 0 & \text{if } 0 < x < a \text{ and } 0 < y < b; \\ u(0,y) = 0 \\ u_x(a,y) = 0 \\ u_y(x,b) = 0 \\ u_y(x,0) = h(x). \end{cases}$$

We seek separated solutions

$$u(x,y) = X(x)Y(y).$$

Substituting into $\Delta u = 0$, we get

$$X''Y + XY'' = 0.$$

Dividing by XY,

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

The homogeneous boundary conditions in x are $X(0) = 0$ and $X'(a) = 0$. Thus,

$$X'' = -\lambda X.$$

The eigenfunctions are

$$X_n(x) = \sin(\beta_n x) \quad \text{where} \quad \beta_n = \frac{\left(n - \frac{1}{2}\right)\pi}{a} \text{ for } n \in \mathbb{N}.$$

The Y-equation is

$$Y'' = \beta_n^2 Y.$$

Thus

$$Y_n(y) = C_n e^{\beta_n y} + D_n e^{-\beta_n y}.$$

The homogeneous condition $Y_n'(b) = 0$ gives

$$C_n \beta_n e^{\beta_n b} - D_n \beta_n e^{-\beta_n b} = 0.$$

Hence $D_n = C_n e^{2\beta_n b}$. Therefore,

$$Y_n(y) = C_n \left(e^{\beta_n y} + e^{2\beta_n b - \beta_n y} \right).$$

Thus

$$u(x,y) = \sum_{n=1}^{\infty} C_n \left(e^{\beta_n y} + e^{2\beta_n b - \beta_n y} \right) \sin(\beta_n x).$$

Now use the inhomogeneous boundary condition $u_y(x,0) = h(x)$. We compute

$$u_y(x,0) = \sum_{n=1}^{\infty} C_n \beta_n \left(1 - e^{2\beta_n b} \right) \sin(\beta_n x).$$

Therefore, if

$$h(x) = \sum_{n=1}^{\infty} h_n \sin(\beta_n x),$$

then

$$C_n = \frac{h_n}{\beta_n (1 - e^{2\beta_n b})}.$$

Using orthogonality,

$$h_n = \frac{2}{a} \int_0^a h(x) \sin(\beta_n x) dx.$$

If a Laplace problem has inhomogeneous boundary conditions on several sides, we split it into simpler problems. For example, suppose $u = u_1 + u_2 + u_3 + u_4$ where each u_i solves Laplace's equation and carries only one of the inhomogeneous boundary conditions, while all other boundary conditions are homogeneous. Since the Laplace equation is linear,

$$\Delta u = \Delta u_1 + \Delta u_2 + \Delta u_3 + \Delta u_4 = 0.$$

The boundary conditions add to give the original boundary data.

We then discuss Laplace's equation on a cube.

6.6 Laplace's Equation on a Cube

Let

$$D = \{0 < x <, \ 0 < y <, \ 0 < z <\}.$$

Consider

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u(,y,z) = g(y,z), & \\ u = 0, & \text{on the other sides.} \end{cases}$$

Seek

$$u(x,y,z) = X(x)Y(y)Z(z).$$

Substituting into $\Delta u = 0$ gives

$$X''YZ + XY''Z + XYZ'' = 0.$$

Dividing by XYZ,

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

The zero boundary conditions in y and z give

$$Y(0) = Y() = 0, \qquad Z(0) = Z() = 0.$$

Hence

$$Y_m(y) = \sin(my), \qquad Z_n(z) = \sin(nz),$$

with eigenvalues

$$m^2, \qquad n^2.$$

Then X satisfies

$$X'' - (m^2 + n^2)X = 0.$$

The condition $u(0,y,z) = 0$ gives

$$X(0) = 0.$$

Thus

$$X_{mn}(x) = \sinh\left(\sqrt{m^2+n^2}\,x\right).$$

Therefore,

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh\left(\sqrt{m^2+n^2}\,x\right)\sin(my)\sin(nz).$$

Using

$$u(,y,z) = g(y,z),$$

we require

$$g(y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh\left(\sqrt{m^2+n^2}\right)\sin(my)\sin(nz).$$

Write

$$g(y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn} \sin(my) \sin(nz).$$

Then

$$A_{mn} = \frac{g_{mn}}{\sinh\left(\sqrt{m^2+n^2}\right)},$$

where

$$g_{mn} = \frac{4}{2} \int_0^{\int} {}^{g}_{0}(y,z) \sin(my) \sin(nz) dydz.$$

Hence

$$u(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn} \frac{\sinh\left(\sqrt{m^2+n^2}\, x\right)}{\sinh\left(\sqrt{m^2+n^2}\right)} \sin(my) \sin(nz).$$

6.7 Laplace's Equation on a Disk

Consider the unit disk

$$B_1 = \{(x,y) : x^2 + y^2 < 1\}.$$

We solve

$$\begin{cases} \Delta u = 0, & \text{in } B_1, \\ u = h(\theta), & \text{on } \partial B_1. \end{cases}$$

In polar coordinates,

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

Seek separated solutions

$$u(r,\theta) = R(r)\Theta(\theta).$$

Then

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0.$$

Multiplying by $r^2/(R\Theta)$,

$$\frac{\Theta''}{\Theta} = -\frac{r^2 R'' + rR'}{R}.$$

Both sides must equal a constant. Since Θ is 2-periodic, we get

$$\Theta'' = -n^2\Theta, \qquad n = 0, 1, 2, \ldots.$$

Thus

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta).$$

The radial equation is

$$r^2 R'' + rR' - n^2 R = 0.$$

For $n \geq 1$, the solutions are

$$R(r) = Cr^n + Dr^{-n}.$$

For $n = 0$, the solutions are

$$R(r) = C + D\log r.$$

Inside the unit disk, we require boundedness at $r = 0$. Therefore, we discard

$$\log r \quad \text{and} \quad r^{-n}.$$

Hence the harmonic function has the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

The boundary condition gives

$$h(\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right].$$

Thus A_n, B_n are the usual Fourier coefficients of h.

6.8 Poisson Formula for the Disk

For the disk

$$B_a = \{x^2 + y^2 < a^2\},$$

the solution of

$$\begin{cases} \Delta u = 0, & \text{in } B_a, \\ u = h(\theta), & \text{on } \partial B_a \end{cases}$$

is given by Poisson's formula:

$$u(r,\theta) = \frac{a^2 - r^2}{2} \int_0^2 \frac{h(\varphi)}{a^2 - 2ar\cos(\theta - \varphi) + r^2} d\varphi.$$

Equivalently, if $x \in B_a$, then

$$u(x) = \frac{a^2 - |x|^2}{2a} \int_{|x'|=a} \frac{u(x')}{|x - x'|^2} dS'.$$

Poisson's formula expresses the value of a harmonic function inside a disk entirely in terms of its boundary values.

6.9 General Harmonic Functions in Polar Coordinates

For a disk, annulus, or exterior domain, the general separated solution of

$$\Delta u = 0$$

in polar coordinates is

$$u(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos(n\theta) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin(n\theta).$$

The boundedness requirement depends on the domain.

- For an annulus $R_1 < r < R_2$, all terms may occur.
- For the interior disk $0 \le r < R$, discard $\log r$, $r^{-n}\cos(n\theta)$, and $r^{-n}\sin(n\theta)$.
- For the exterior domain $r > R$, terms growing at infinity are usually discarded when boundedness or decay at infinity is required.

6.10 Separation of Variables on a Wedge

Consider the wedge

$$0 < r < a, \qquad 0 < \theta < \beta.$$

Solve

$$\Delta u = 0$$

with zero Dirichlet boundary conditions on the straight sides:

$$u(r,0) = u(r,\beta) = 0,$$

and inhomogeneous Neumann condition on the curved side:

$$\frac{\partial u}{\partial r}(a,\theta) = h(\theta).$$

Use the ansatz

$$u(r,\theta) = R(r)\Theta(\theta).$$

The separated equations are

$$\Theta'' + \lambda\Theta = 0,$$

and

$$r^2R'' + rR' - \lambda R = 0.$$

The boundary conditions

$$\Theta(0) = \Theta(\beta) = 0$$

give

$$\Theta_n(\theta) = \sin\left(\frac{n\theta}{\beta}\right), \qquad n = 1,2,3,\ldots.$$

Thus

$$\lambda_n = \left(\frac{n}{\beta}\right)^2.$$

The radial equation becomes

$$r^2R'' + rR' - \left(\frac{n}{\beta}\right)^2 R = 0.$$

Thus

$$R_n(r) = C_n r^{n/\beta} + D_n r^{-n/\beta}.$$

Since $r = 0$ lies in the wedge, boundedness gives

$$D_n = 0.$$

Hence

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n/\beta} \sin\left(\frac{n\theta}{\beta}\right).$$

Now impose

$$u_r(a,\theta) = h(\theta).$$

We compute

$$u_r(a,\theta) = \sum_{n=1}^{\infty} A_n \frac{n}{\beta} a^{n/\beta-1} \sin\left(\frac{n\theta}{\beta}\right).$$

Expand

$$h(\theta) = \sum_{n=1}^{\infty} h_n \sin\left(\frac{n\theta}{\beta}\right),$$

where

$$h_n = \frac{2}{\beta}\int_0^{\beta} h(\theta)\sin\left(\frac{n\theta}{\beta}\right) d\theta.$$

Then

$$A_n = \frac{h_n}{\frac{n}{\beta}a^{n/\beta-1}}.$$

6.11 Mean Value Property

Theorem 6.2 (Mean value property). Let $\Omega \subseteq \mathbb{R}^n$ be open, and suppose

$$u \in C^2(\Omega), \qquad \Delta u = 0 \quad \text{in } \Omega.$$

If

$$B_r(x_0) \subseteq \Omega,$$

then

$$u(x_0) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x_0)} u(y)dS,$$

and

$$u(x_0) = \frac{1}{|B_r|}\int_{B_r(x_0)} u(y)dy.$$

The first formula says that the value of a harmonic function at the center of a ball is the average of its values over the sphere. The second formula says that it is also the average over the whole ball.

6.11.1 Proof sketch

By translating, assume $x_0 = 0$. Define

$$F(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} u(y)dS.$$

Writing $y = r\omega$, where $\omega \in \partial B_1$, we have

$$F(r) = \frac{1}{|\partial B_1|}\int_{\partial B_1} u(r\omega)dS_\omega.$$

Differentiating,

$$F'(r) = \frac{1}{|\partial B_1|}\int_{\partial B_1} \nabla u(r\omega)\cdot\omega dS_\omega.$$

Changing back to ∂B_r,

$$F'(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} \frac{\partial u}{\partial n} dS.$$

By the divergence theorem,

$$\int_{\partial B_r} \frac{\partial u}{\partial n} dS = \int_{B_r} \Delta u dx = 0.$$

Thus

$$F'(r) = 0.$$

Hence $F(r)$ is constant in r. Letting $r \to 0$, continuity gives

$$F(r) = u(0).$$

This proves the spherical mean value property. The ball version follows by integrating the spherical version over $0 \le \mathbb{R}ho \le r$.

Example 6.1. Let D be the unit disk centered at $(0,0)$. If u is harmonic in D and

$$u(0,0) = 1,$$

then by the ball mean value property,

$$1 = \frac{1}{|D|}\int_D u(x)dx.$$

Since $|D| =$, we get

$$\int_D u(x)dx = .$$

6.12 Strong Maximum Principle

Theorem 6.3 (Strong maximum principle). Let $D \subseteq \mathbb{R}^n$ be connected and bounded. Suppose

$$u \in C^2(D) \cap C(\overline{D}), \qquad \Delta u = 0 \quad \text{in } D.$$

If u attains its maximum at an interior point of D, then u is constant in D.

Proof. Let

$$M = \max_{\overline{D}} u.$$

Suppose $u(x_0) = M$ for some $x_0 \in D$. Choose a ball

$$B_r(x_0) \subseteq D.$$

By the mean value property,

$$M = u(x_0) = \frac{1}{|B_r|}\int_{B_r(x_0)} u(y)dy.$$

But $u(y) \le M$ everywhere in $B_r(x_0)$. Therefore the average can equal M only if

$$u(y) = M$$

throughout $B_r(x_0)$.

Thus the set where $u = M$ is open. It is also closed by continuity. Since D is connected, this set is all of D. Hence $u \equiv M$. □

6.13 Liouville's Theorem

Theorem 6.4 (Liouville's theorem for harmonic functions). If u is harmonic in all of $\mathbb{R}^n$ and

$$|u(x)| \leq C \qquad \text{for all } x \in \mathbb{R}^n,$$

then u is constant.

6.13.1 Idea of proof in dimension two

Let $p, q \in \mathbb{R}^2$. Choose a large disk $B_R(p)$. By the mean value property,

$$u(p) = \frac{1}{|B_R(p)|} \int_{B_R(p)} u(y)dy.$$

Similarly,

$$u(q) = \frac{1}{|B_R(q)|} \int_{B_R(q)} u(y)dy.$$

For large R, the disks $B_R(p)$ and $B_R(q)$ almost overlap. The difference between their averages is bounded by the contribution from the symmetric difference of the disks. Since $|u| \leq C$, this difference tends to 0 as $R \to \infty$. Hence

$$u(p) = u(q).$$

Since p, q were arbitrary, u is constant.

6.14 Green's First Identity

Let $D \subseteq \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary. Recall the product rule:

$$\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u\Delta v.$$

By the divergence theorem,

$$\int_D \nabla \cdot (u\nabla v)dx = \int_{\partial D} u\frac{\partial v}{\partial n}dS.$$

Therefore,

$$\int_{\partial D} u\frac{\partial v}{\partial n}dS = \int_D \nabla u \cdot \nabla v dx + \int_D u\Delta v dx.$$

Theorem 6.5 (Green's first identity). For sufficiently smooth u, v,

$$\int_{\partial D} v\frac{\partial u}{\partial n}dS = \int_D \nabla v \cdot \nabla u dx + \int_D v\Delta u dx.$$

6.15 Green's Second Identity

Interchanging u and v in Green's first identity gives

$$\int_{\partial D} u\frac{\partial v}{\partial n}dS = \int_D \nabla u\cdot\nabla v dx + \int_D u\Delta v dx.$$

Subtracting the two identities, we obtain:

Theorem 6.6 (Green's second identity). For sufficiently smooth u, v,

$$\int_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = \int_D (u\Delta v - v\Delta u)\, dx.$$

6.16 Fundamental Solution of Laplace's Equation

The fundamental solution of the Laplace equation is a radial function satisfying

$$\Delta(x) = 0 \qquad \text{for } x \neq 0,$$

and distributionally,

$$\Delta = \delta_0.$$

For $n = 2$,

$$(x) = \frac{1}{2}\log|x|.$$

For $n \geq 3$,

$$(x) = -\frac{1}{(n-2)|\partial B_1|}\frac{1}{|x|^{n-2}}.$$

Equivalently,

$$(x) = \frac{1}{(2-n)|\partial B_1|}|x|^{2-n}.$$

Theorem 6.7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^1 and compactly supported. Define

$$u(x) = (*f)(x) = \int_{\mathbb{R}^n} (x-y)f(y)dy.$$

Then

$$\Delta u = f \qquad \text{in } \mathbb{R}^n.$$

6.16.1 Idea of proof

Formally,

$$\Delta u = \Delta(*f) = (\Delta) * f = \delta_0 * f = f.$$

To justify this, one removes a small ball around the singularity, integrates by parts, and then lets the radius of the small ball tend to zero. The boundary integral over the small sphere converges to $f(x)$, while the remaining terms vanish.

6.17 Representation Formula for Harmonic Functions

Let u be harmonic in D, and fix $x_0 \in D$. Take

$$v(x) = (x - x_0).$$

Then

$$\Delta u = 0,$$

and distributionally,

$$\Delta v = \delta_{x_0}.$$

Using Green's second identity gives the representation formula

$$u(x_0) = \int_{\partial D} \left[u(x) \frac{\partial}{\partial n}(x - x_0) - (x - x_0) \frac{\partial u}{\partial n}(x) \right] dS(x).$$

This formula represents a harmonic function inside D using boundary data. However, it involves both u and $\frac{\partial u}{\partial n}$ on the boundary, so it is not directly a solution formula for pure Dirichlet or pure Neumann problems.

6.18 Another Proof of the Mean Value Property

Let u be harmonic in $B_R(x_0)$. Apply the representation formula to the ball $D = B_R(x_0)$. On the boundary $\partial B_R(x_0)$, the fundamental solution $(x - x_0)$ is constant, since $|x - x_0| = R$. Since

$$\int_{B_R(x_0)} \Delta u dx = 0,$$

we have

$$\int_{\partial B_R(x_0)} \frac{\partial u}{\partial n} dS = 0.$$

Thus the term involving $\frac{\partial u}{\partial n}$ vanishes. The remaining term gives

$$u(x_0) = \frac{1}{|\partial B_R|} \int_{\partial B_R(x_0)} u(x) dS.$$

This is the spherical mean value property.

6.19 Green Function and Poisson Kernel

We want to solve

$$\begin{cases} \Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

A solution formula is expected to have the form

$$u(x) = \int_\Omega f(y) G(x,y) dy + \int_{\partial\Omega} g(y) P(x,y) dS(y).$$

Here:

- $G(x,y)$ is the Green function of the domain Ω;
- $P(x,y)$ is the Poisson kernel of the domain Ω.

6.19.1 Green function

For fixed $x \in \Omega$, the Green function $G(x,y)$ is constructed as

$$G(x,y) = (x-y) -^{x} (y),$$

where ^{x} is harmonic in Ω and chosen so that

$$G(x,y) = 0 \qquad \text{for } y \in \partial\Omega.$$

Thus G has the same singularity as the fundamental solution, but satisfies zero boundary condition.
Using Green's identity with G, one obtains

$$u(x) = \int_{\Omega} f(y)G(x,y)dy + \int_{\partial\Omega} g(y)\frac{\partial G}{\partial n_y}(x,y)dS(y),$$

up to the sign convention chosen for the normal derivative and the fundamental solution.
Thus the Poisson kernel is essentially the normal derivative of the Green function:

$$P(x,y) = \frac{\partial G}{\partial n_y}(x,y).$$

Bibliography

[1] Lawrence C. Evans, *Partial Differential Equations*, 2nd ed., Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 2010. ISBN: 978-0-8218-4974-3.

[2] Walter A. Strauss, *Partial Differential Equations: An Introduction*, 2nd ed., John Wiley & Sons, Hoboken, NJ, 2008. ISBN: 978-0-470-05456-7.