# MA3264 Mathematical Modelling

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These notes are based off **Prof. Brett Mcinnes's** MA3264 Mathematical Modelling materials.

This set of notes was last updated on February 24, 2026. If you would like to contribute a nice discussion to the notes or point out a typo, please send me an email at thangpangern@u.nus.edu.

# Contents

Chapter 1

# Models using First-Order Differential Equations

## 1.1 Modelling with Separable Differential Equations

Mathematical models are rarely realistic. So, what is their purpose? The value lies in their ability to evolve. When a model predicts something nonsensical, it highlights where the simplifications fall short. You refine the model, adding complexity to better align it with reality. This iterative process mirrors life itself: start simple, and work towards something more realistic.

Take $\sin\theta = \theta$ as an example. Is it accurate? No, but it's a reasonable approximation for small $\theta$. To improve it, we can use $\sin\theta = \theta - \theta^3/6$. Is that entirely true? Not quite, but it is closer to the truth. And with further refinements, we can get even better approximations.

> Definition 1.1 (separable DE). A first-order differential equation is separable if it can be written as
>
> $$M(x)\ dx = N(y)\ dy.$$
>
> Notice that in this form, we say that we have *separated the variables* as everything involving $x$ is on one side and everything involving $y$ is on the other.

One should know how to solve such differential equations as shown in Definition 1.1 from one's high school days (or even MA2002 Calculus) — simply integrate both sides, i.e.

$$\int M(x)\ dx = \int N(y)\ dy + c.$$

Example 1.1 (radioactive decay). Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with a sample containing 2 mg of this substance at certain time, say $t = 0$, what can be said about the amount available at a later time?

*Solution.* There could be a variety of radioactive materials present, and some of them might contribute to generating the substance we are analysing. As such, we shall deliberately disregard all other materials. So,

$$\frac{dm}{dt} = -km$$

where $m(t)$ is the amount of substance at time $t$, so $m(0) = 2$. Also, $k$ is some arbitrary constant. With some algebraic manipulation, we have

$$\frac{1}{m}\,dm = -k\,dt.$$

Note that this admits the form in Definition 1.1, so one can integrate both sides to obtain

$$\ln\left(\frac{m}{c}\right) = -kt \quad \text{which implies} \quad m = ce^{-kt}.$$

Here, $c$ is also an arbitrary constant. Setting $m(0) = 2$, we have $c = 2$, so $m = 2e^{-kt}$ — anyway, this means that the radioactive substance will decay at an exponential rate. This process is commonly known as an example of exponential decay. □

Example 1.2 (black holes). Stephen Hawking discovered that black holes lose mass over time, in addition to gaining it through the process of accreting matter. He developed a model to describe this complex phenomenon by simplifying the situation: he disregarded the details of matter falling into the black hole and concentrated solely on the radiation emitted, known as Hawking radiation. The rate of mass loss is described by the following differential equation:

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2}$$

where $t$ is time, $M$ is the mass of the black hole, $\hbar$ is the reduced Planck's constant, $c$ is the speed of light, and $G$ is the universal gravitational constant.

One can easily compute the time $T$ it takes for a black hole to disappear completely, i.e. the lifetime of a black hole with initial mass $M_0$. We have

$$T = \frac{5120\pi G^2 M_0^3}{\hbar c^4}.$$

Example 1.3 (planetary orbit). The orbit of a planet represents the path it follows as it moves around the Sun. In reality, this trajectory is highly complex due to gravitational influences from other planets, which pull on it from various directions. Isaac Newton, however, devised a simplified model of this situation by focusing exclusively on the interaction between the Sun and a single planet. He ignored the effects of other planets, asteroids, and miscellaneous items, as well as the fact that the Sun is not a perfect sphere, among other complexities. This approach allowed him to derive foundational insights into planetary motion.

In order to understand Newton's model of planetary orbits, one needs to recall polar coordinates (recall from MA2104)! Using his laws of motion, Newton discovered that a planet in his model has an orbit which satisfies the differential equation

$$\left(\frac{du}{d\theta}\right)^2+(u-A)^2=B^2,$$

where $u(\theta)=1/r(\theta)$ and $A,B>0$ are constants with $B/A<1$. Note that $r(\theta)$ is the equation of our graph in polar coordinates.

This differential equation is separable, i.e. one can show that

$$d\theta=\frac{du/B}{\sqrt{1-\left(\frac{u-A}{B}\right)^2}}.$$

Integrating both sides yields

$$\theta+c=\arcsin\left(\frac{u-A}{B}\right).$$

Since $u=1/r$, it follows that

$$r=\frac{1/A}{1+\frac{B}{A}\sin(\theta+c)}.$$

Since $B/A<1$, we would see that this curve looks like an ellipse[1]! As such, in this simplified model of the solar system, all the planets have elliptical orbits (also known as Kepler's first law of planetary motion).

Example 1.4 (MA3264 AY25/26 Sem 1 Tutorial 1). Solve the equation $y'=y$, $y(0)=1$, in the following way: assume that $y$ has an expansion of the form

$$y=a_0+a_1x+a_2x^2+\ldots \tag{1.1}$$

and use the equation and the initial conditions to find the numbers $a_n$ for all $n$. Next, consider the equation

$$y'=2\sqrt{y}\quad\text{where}\quad y(x)\geq 0\text{ and }y(0)=0.$$

The previous method doesn't work. So find the solution in some other way.

*Solution.* This is a simple exercise that is also covered in MA3220 Ordinary Differential Equations. Suppose $y$ admits the power series solution as in (1.1). Then,

$$y'=a_1+2a_2x+3a_3x^2+\ldots.$$

Comparing the coefficients, we have $a_0=a_1$, $2a_2=a_1$, $3a_3=a_2$ and so on. Suppose $a_0=c$, where $c$ is an arbitrary constant. Then, $a_1=c$, $a_2=c/2$, $a_3=c/3!$ and so on. In general,

$$a_n=\frac{c}{n!}\quad\text{so}\quad y=\sum_{n=0}^{\infty}a_nx^n=c\sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{1.2}$$

[1] Given the range of values of the ratio $B/A$, we can obtain various conic sections.

Substituting the initial condition $y(0) = 1$ yields $c = 1$, so $y = e^x$ (recall that the infinite series obtained on the right side of (1.2) represents $e^x$).

On the other hand, for the differential equation $y' = 2\sqrt{y}$, a brief modification we can make is to let

$$\sqrt{y} = b_0 + b_1 x + b_2 x^2 + \ldots. \tag{1.3}$$

Squaring both sides and realising that the product of two infinite series can be interpreted as a convolution (search *Cauchy product*), we have

$$y = \left(\sum_{m=0}^{\infty} b_m x^m\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} b_i b_{k-i}\right) x^k.$$

Upon expansion, we have

$$y = b_0^2 + 2b_0 b_1 x + (2b_0 b_2 + b_1^2)x^2 + (2b_0 b_3 + 2b_1 b_2)x^3 + \ldots$$

so

$$y' = 2b_0 b_1 + 2\left(2b_0 b_2 + b_1^2\right)x + 3\left(2b_0 b_3 + 2b_1 b_2\right)x^2 + \ldots. \tag{1.4}$$

Since $y' = 2\sqrt{y}$, by comparing the coefficients in (1.3) and (1.4), we have $2b_0 b_1 = 2b_0$, $4b_0 b_2 + 2b_1^2 = 2b_1$, $6b_0 b_3 + 6b_1 b_2 = 2b_2$ and so on. Since $y(0) = 0$, then $b_0^2 = 0$, so $b_0 = 0$. From $2b_0 b_1 = 2b_0$, we have $b_1 = 1$ or $b_0 = 0$. Hence, $b_1 = 1$. From $6b_0 b_3 + 6b_1 b_2 = 2b_2$, we have $6b_2 = 2b_2$, so $b_2 = 0$. One can then deduce that $b_n = 0$ for all $n \geq 3$. We conclude that $y = x^2$ and $y = 0$ are solutions to the second differential equation. Of course, we could have proceeded with the usual technique of separating the variables and integrating both sides. □

Example 1.5 (MA3264 AY25/26 Sem 1 Tutorial 1). One theory about the behaviour of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near to a candle and mistakes it for the Moon. What will happen to the moth?

Note that in polar coordinates $(r, \theta)$, the formula for the angle $\psi$ between the radius vector and the velocity vector is given by

$$\tan\psi = r\frac{d\theta}{dr}.$$

If you wish to derive this formula[2], recall that the tangential component of a small displacement in polar coordinates $(r, \theta) \mapsto (r + dr, \theta + d\theta)$ is $r\,d\theta$ and the radial component is just $dr$. Use the formula to solve for $r$ as a function of $\theta$.

[2] As the point moves by an infinitesimal amount $(dr, d\theta)$, the displacement vector in Cartesian space is composed of a radial component of length $dr$ and a tangential component of $r\,d\theta$. Hence, the tangent of the angle $\psi$ between the radius vector and the velocity vector is given by the quotient of the opposite length with the adjacent length, which is $r\,d\theta/dr$.

*Solution.* We have

$$\frac{1}{r}\,dr = \cot\psi\,d\theta.$$

Then, integrate both sides. Solving this differential equation gives a logarithmic spiral of equation

$$r(\theta) = r_0 e^{\theta\cot\psi}.$$

A logarithmic spiral winds inexorably toward (or away from) the origin depending on the sign of $\cot\psi$. Since the moth is attempting to approach the candle, it traces the part of the logarithmic spiral that converges towards the candle. □

## 1.2 Modelling with Linear Differential Equations

Example 1.6 (melting ice). The Arctic Ocean (Figure 1.1) plays a crucial role in the global climate system, as it is the region most impacted by global warming. It is warming at approximately three times the rate of the rest of the planet, with the pace continuing to accelerate.

Figure 1.1: The Arctic Ocean

The surface of the Arctic Ocean consists of both ice-covered areas and open water. Let $I(t)$ represent the area covered by ice and $W(t)$ the area of open water, both as functions of time. The temperature $T(t)$ is also time-dependent. The rate of change of the ice-covered area, $I(t)$, is negatively influenced by the temperature $T(t)$, while the rate of change of the temperature is positively affected by $W(t)$. As such, we obtain the following pair of differential equations:

$$\frac{dI}{dt} = -aT \quad \text{and} \quad \frac{dT}{dt} = bW \quad \text{where } a,b>0 \text{ are constants}$$

This relationship arises because ice, being highly reflective (a property known as having high albedo), reflects most sunlight, preventing it from absorbing significant heat. In contrast, open water, which appears dark blue or nearly black, absorbs heat efficiently. Consequently, when $W(t)$ is large, more heat is absorbed, causing the temperature to rise.

The equations are called *linear* because of the absence of terms like $T^3$ or $\cos W$ — we only see $T$ and $W$. The trick to solving such a pair of simultaneous equations is to differentiate one of the equations. In particular, we differentiate the first equation and substitute into the second one to obtain

$$\frac{d^2 I}{dt^2} = -abW.$$

We note that the total area of the Arctic Ocean is a constant, which is equal to $I + W$. Differentiating a constant twice yields 0, so

$$\frac{d^2 I}{dt^2} + \frac{d^2 W}{dt^2} = 0 \quad \text{which implies} \quad \frac{d^2 W}{dt^2} = abW.$$

We will learn how to solve such differential equations in Chapter 2. Anyway, one checks that

$$W(t) = Ae^{\sqrt{ab}t} + Be^{-\sqrt{ab}t} \quad \text{satisfies the differential equation.}$$

Here, $A$ and $B$ are some constants. Unless $A = 0$, the expression will blow up exponentially fast, with $W$ increasing rapidly till it reaches the total area, and there will not be any ice at all. In fact it is feared that exactly this will happen some time this century!

Of course, this hasn't happened (yet), which suggests that there is a potential flaw in our model. However, nothing is actually wrong since it is just a model after all! The Arctic Ocean is an extraordinarily complex system governed by hundreds, if not thousands, of parameters and interrelated processes. Nevertheless, we need to begin somewhere. Having said that, see Example 1.8 for a modified setup.

Example 1.7 (cosmology; MA3264 AY25/26 Sem 1 Tutorial 1). In Cosmology, the ratio of the sizes of the Universe at two different times is measured by a function of time called the *scale function*, denoted by $a(t)$. What are the units of $a(t)$?

The Friedmann equation relates this function to the energy density of the Universe and to its spatial curvature. In a particular cosmological model, the Friedmann equation takes the form

$$L^2 \dot{a}^2 = a^2 - \frac{2}{a^2} + 1 \quad \text{where} \quad \dot{a} = \frac{da}{dt} \text{ denotes the time derivative.} \tag{1.5}$$

Also, $L$ denotes a positive constant, and the initial condition is $a(0) = 1$. What are the units of $L$? Show, without solving this equation, that the universe described by this model is never smaller than a certain minimum size. Now solve the equation and describe the history of this universe.

*Solution.* First, note that $a(t)$ is dimensionless. We then solve the Friedmann differential equation (1.5). We have

$$L^2\left(\frac{da}{dt}\right)^2 = \frac{\left(a^2+2\right)\left(a^2-1\right)}{a^2} \quad \text{so} \quad \frac{da}{dt} = \frac{1}{L}\cdot\frac{\sqrt{\left(a^2+2\right)\left(a^2-1\right)}}{a}.$$

One can deduce that the minimum value of $a$ is 1. Then, one can use the substitution $u=\sqrt{a^2+2}$ to integrate, which yields

$$\frac{t}{L} = \ln\left(\sqrt{a^2+2}+\sqrt{a^2-1}\right)+c.$$

When $t=0$, $a=1$, so $c=\frac{1}{2}\ln 3$. Making $a$ the subject of the equation, we have

$$a(t) = \frac{1}{\sqrt{2}}\cdot\sqrt{3\cosh\left(\frac{2t}{L}\right)-1}.$$

At $t=0$, we have $a=1$ and $\dot{a}=0$. For $t<0$, we have $\dot{a}<0$ (contraction) down to the minimum $a_{\min}=1$. Lastly, for $t>0$, $\dot{a}>0$ which denotes expansion. □

Example 1.8 (modified melting ice; MA3264 AY25/26 Sem 1 Tutorial 1). In Example 1.6, we constructed a model of Arctic sea ice using the equations (all parameter values have been chosen just for convenience)

$$\frac{dI}{dt} = -T \quad \text{and} \quad \frac{dT}{dt} = \sin(5t)\,W, \tag{1.6}$$

where $I$ denotes the area of the ice and $W$ denotes the area of open water. The $\sin 5t$ represents seasonal variations. Notice that $\sin 5t$ is sometimes negative. This is because when the atmospheric weather gets hot enough, the water (which is always pretty cold) actually helps to lower temperatures. Argue that, if the total area of the Arctic Ocean is 10 in these units, then this is a model of the fluctuating area of the open sea in pre-industrial times. Notice that there is a long-term variation as well as the expected seasonal one.

Now the Industrial Revolution happens and carbon dioxide is emitted into the atmosphere, causing a slow global rise in temperatures. Let us model this with the equations

$$\frac{dI}{dt} = -T \quad \text{and} \quad \frac{dT}{dt} = \sin(5t)\,W+0.01t. \tag{1.7}$$

What does your model predict now?

*Solution.* We begin by discussing the first pair of differential equations (1.6). Recall that $I$ denotes the area of the ice and $W$ denotes the area of the open water. Suppose $I+W=10$. Clearly,

$$\frac{dI}{dt}+\frac{dW}{dt}=0 \quad \text{so} \quad \frac{dW}{dt} = -\frac{dI}{dt} = T.$$

Hence,

$$\frac{d^2W}{dt^2} = \frac{dT}{dt} = W\sin 5t.$$

That is, the model describes seasonal oscillations superimposed on a possible slow (long-term) drift set by the initial heat content. As for the second model (1.7), the added $0.01t$ term imposes an accelerating warming, driving a superlinear increase in open water and eventual ice-free conditions, with the seasonal cycle riding on an ever-rising trend. □

First-order linear ODEs are very useful. However, they are not always separable. Having said that, there is a trick that allows us to solve them.

**Proposition 1.1 (integrating factor).** Consider linear differential equations of the form

$$\frac{dy}{dx} + yP(x) = Q(x),$$

where $P$ and $Q$ are functions of $x$. One can solve such differential equations by multiplying both sides of the equation by an integrating factor $\mu(x)$, then use the product rule, where

$$\mu(x) = \exp\left(\int P(t)\ dt\right).$$

Proposition 1.1 has already been covered in MA2002 so we will not discuss further. Now, what happens if the differential equation is neither separable nor linear? One nice instance is when we come across a Bernoulli equation (Definition 1.2).

**Definition 1.2 (Bernoulli equation).** The differential equation

$$\frac{dy}{dx} + yP(x) = Q(x)y^n,$$

where $n \in \mathbb{R}$, is a Bernoulli equation.

Again, Definition 1.2 has already been covered in MA2002 — the trick to solving such equations is to introduce the substitution $z = y^{1-n}$.

**Example 1.9 (Bernoulli equation; MA3264 AY25/26 Sem 1 Tutorial 2).** Solve the differential equation

$$2xy\frac{dy}{dx} + (x-1)y^2 = x^2e^x.$$

*Solution.* We have

$$\frac{dy}{dx} + \left(\frac{x-1}{2x}\right)y = \frac{1}{2}xe^xy^{-1}.$$

This is a Bernoulli equation (Definition 1.2). We use the substitution $z = y^2$ and omit the remaining details. □

**Example 1.10 (mixing problem).** At time $t = 0$, a tank contains 2 kg of salt dissolved in 100 $\ell$ of water. Assuming that the water containing 0.25 kg of salt per litre is entering the tank at a rate of 3 $\ell$/min and the well-stirred solution is leaving the tank at the same rate. Find the amount of salt at any time $t$. Again, such questions have already been discussed in MA2002 so we will skip.

**Example 1.11 (mixing problem).** Imagine an experiment where a planet with a pristine atmosphere begins receiving 50 billion tons of $CO_2$ annually. The $CO_2$ mixes uniformly with the air, while biological and geological processes remove it, keeping the total atmospheric volume nearly constant. Based on what we have discussed thus far, the concentration of $CO_2$ would rise exponentially toward a limiting value.

Warned by their scientists, the planet's inhabitants immediately reduce the $CO_2$ concentration in their emissions at a rate inversely proportional to time.

Now, consider the following analogous problem. A tank contains $100\ \mathrm{m}^3$ of pure air (negligible $CO_2$) at $t = 1$ second. At that moment, polluted air with a $CO_2$ concentration of $10/t\ \ell/\mathrm{m}^3$ starts flowing in at $10\ \mathrm{m}^3/\mathrm{s}$. The mixture in the tank is pumped out at the same rate. Plot the quantity of $CO_2$ in the tank as a function of time.

*Solution.* We have

$$\frac{dQ}{dt} = \frac{100}{t} - \frac{Q}{10} \quad \text{with initial condition } Q(1) = 0.$$

The integrating factor is $e^{t/10}$ so we obtain

$$Q(t) = 100e^{-t/10}\left[\mathrm{Ei}\left(\frac{t}{10}\right) - \mathrm{Ei}\left(\frac{1}{10}\right)\right].$$

Here, $\mathrm{Ei}(x)$ denotes the exponential integral (Definition 1.3).

**Definition 1.3 (exponential integral).** For real non-zero values of $x$, define $\mathrm{Ei}(x)$ to be

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt.$$

One can plot the graph of $Q(t)$. We see that the amount of $CO_2$ in the tank increases for some time even though the concentration in the gas entering the tank is decreasing. It reaches a rather high maximum, before decreasing rather slowly. This is known as the dreaded *momentum effect*, i.e. even if we start drastic reductions of $CO_2$ release now, the amount of it in the atmosphere will increase for a long time and will only be reduced to safe levels in the distant future[3]. □

**Example 1.12 (mixing problem; MA3264 AY16/17 Sem 1 Quiz 1).** A tank contains, at time $t = 1$ second, 100 cubic metres of pure water. At that instant, salty water with a concentration of 10 kilograms per cubic metre begins to be pumped in at a rate of $10/t$ cubic metres per second, and the mixed solution is pumped out at the same rate. How many kilograms of salt are there in the tank after $10^{10}$ seconds?

*Solution.* Let $m(t)$ denote the mass of salt at time $t$. Suppose the experiment starts at $t = 1$ so $m(1) = 0$. By considering rate in – rate out, we have

$$\frac{dm}{dt} = 10 \cdot \frac{10}{t} - \frac{m}{100} \cdot \frac{10}{t} = \frac{100}{t} - \frac{m}{10t}.$$

We rearrange to obtain

$$\frac{dm}{dt} + m\left(\frac{1}{10t}\right) = \frac{100}{t}.$$

Use the integrating factor method, one can show that $m\left(10^{10}\right) = 900$. □

[3] One can look up 'representative concentration pathway' for more notes on this

Example 1.13 (radioactive decay). Sometimes, the product of radioactive decay is itself a radioactive substance that undergoes decay at a different rate. An example is uranium-thorium dating, a method used by paleontologists to estimate the age of fossils, particularly ancient corals.

Corals filter seawater (Figure 1.2), which contains trace amounts of uranium-234, a radioactive isotope. These corals absorb uranium-234 into their skeletons while alive. Over time, uranium-234 decays (to be precise, the type of radioactive decay that uranium-234 goes through is alpha decay) into thorium-230, another radioactive element via the equation

$$^{234}_{92}\mathrm{U} \rightarrow ^{230}_{90}\mathrm{Th} + ^{4}_{2}\alpha.$$

Uranium-234 has a half-life of 245,000 years, while thorium-230 has a shorter half-life of 75,000 years.

Thorium-230 is not naturally present in seawater, so when a coral dies, its skeleton contains uranium-234 but no thorium-230. This is because the coral's lifespan is negligible compared to uranium-234's half-life. However, over time, as uranium-234 decays, thorium-230 begins to accumulate in the coral skeleton. By measuring the ratio of uranium-234 to thorium-230 in a coral sample, we can estimate the time elapsed since the coral's death — its age.

This information is crucial for understanding events like mass coral die-offs. If corals have historically died off regularly over long periods, it might suggest that current coral deaths are part of a natural cycle rather than solely caused by global warming.

Figure 1.2: Corals in the Great Barrier Reef, Australia

To model this process, we make certain simplifying assumptions. Although other radioactive materials may be present, we ignore them because the decay products of thorium-230 typically decay much faster than uranium-234 or thorium-230 itself. Therefore, their contributions are negligible for our purposes.

Let $U(t)$ represent the number of uranium-234 atoms in the coral sample at time $t$, and $T(t)$ represent the number of thorium-230 atoms. Since each uranium-234 atom decay produces one thorium-230 atom, the rate at which thorium-230 is produced equals the rate at which uranium-234 decays. Consequently, we have the following relationships for the decay rates:

$$\frac{dU}{dt} = -k_U U \quad \text{and} \quad \frac{dT}{dt} = k_U U - k_T T, \tag{1.8}$$

where $k_U$ and $k_T$ are constants with $k_U \neq k_T$, and $U(0) = U_0$ and $T(0) = 0$. We wish to find $t$ given that we know the ratio of $T(t)$ to $U(t)$ at the present time. Solving with the given data (first equation) yields

$$U = U_0 e^{-k_U t}.$$

One can attempt to solve for $k_U$ and $k_T$, which are

$$k_U = \frac{\ln 2}{245,000} \quad \text{and} \quad k_T = \frac{\ln 2}{75,000}. \tag{1.9}$$

The second differential equation yields

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}.$$

Solving with $T(0) = 0$ yields

$$T(t) = \frac{k_U}{k_T - k_U} U_0 \left( e^{-k_U t} - e^{-k_T t} \right).$$

Although we do not know the value of $U_0$, we can consider the ratio $T/U$, which is

$$\frac{T}{U} = \frac{k_U}{k_T - k_U} \left[ 1 - e^{(k_U - k_T)t} \right] \tag{1.10}$$

So, if we compute the ratio $T/U$ at the present time, we can solve for $t$ and obtain our answer!

Example 1.14 (radioactive decay; MA3264 AY25/26 Sem 1 Tutorial 2). The half-life of thorium-230 is about 75000 years, while that of uranium-234 is about 245000 years. A certain sample of ancient coral has a thorium/uranium ratio of 10 percent. How old is the coral?

*Solution.* Recall the radioactive differential equations (1.8) discussed in Example 1.13. We also deduced the values of $k_U$ and $k_T$ in (1.9), where 245,000 and 75,000 in the denominators are known as the respective decay constants. In this question, we are given that $T/U = 0.1$. One can substitute the known quantities into (1.10) to obtain the value of $t$, which is approximately 40083 years. □

Example 1.15 (catenary; MA3264 AY25/26 Sem 1 Tutorial 2). If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a *catenary*. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and Australia. For example, the Gateway Arch in Missouri, United States of America, is in the shape of an inverted catenary.

Figure 1.3: The Gateway Arch in Missouri, United States of America

It can be shown using simple physics that if the shape is given by a function $y(x)$, then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\,dt, \tag{1.11}$$

where $x = 0$ at the lowest point of the catenary and $y(0) = 0$, $\mu$ is the weight per unit length of the cable, and $T$ is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. One can use the Fundamental Theorem of Calculus, and think of the resulting equation as a first-order ordinary differential equation.

*Solution.* Differentiating both sides of the catenary differential equation (1.11) yields

$$\frac{d^2y}{dx^2} = \frac{\mu}{T}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{1.12}$$

For compactness, let $c = \mu/T$. Then, squaring both sides of (1.12) yields

$$\left(\frac{d^2y}{dx^2}\right)^2 = c^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right].$$

Using the substitution

$$u = \frac{dy}{dx} \quad \text{we have} \quad \frac{du}{dx} = c\sqrt{u^2+1}.$$

One can use the substitution $u = \tan\theta$ to solve the new differential equation. We omit the details. $\square$

Example 1.16 (potato blight; MA3264 AY25/26 Sem 1 Tutorial 2). The spread of potato blight[4] in a crop (Figure 1.4) can be modelled by the differential equation

$$\frac{dx}{dt} = Kf(t)(1-x),$$

where $x(t)$ denotes the fraction of plants that are infected at time $t$, $K > 0$ is a constant describing the infectivity of the disease, and $p > 0$ and $q > 0$ represent physiological delay parameters. We assume that no plant is infected until $t = 0$, and that at $t = 0$ a fraction $\alpha \in (0,1)$ of susceptible plants suddenly become infected. Also,

$$f(t) = x(t-p) - x(t-p-q).$$

Note that this is a function in terms of $t$!

Figure 1.4: A potato blight

Show that the solution to the differential equation can be written in the form

$$x(t) = 1 - (1-\alpha)\exp\left(-K\int_0^t f(\tau)\,d\tau\right).$$

This expression still does not determine $x(t)$ explicitly since $x(t)$ also appears on the right side of the equation. However, it can be used iteratively as follows:

**(i)** Show that for any $a > 0$,

$$\int_0^t x(\tau - a)\,d\tau = \int_0^{t-a} x(\tau)\,d\tau,$$

interpreting $x(s) = 0$ for all $s < 0$.

**(ii)** Use **(i)** to prove the identity

$$\int_0^t f(\tau)\,d\tau = \int_{(t-p)-q}^{t-p} x(\tau)\,d\tau.$$

[4] A blight is a plant disease, typically one caused by fungi.

**(iii)** Deduce that

$$\lim_{t\to\infty}\int_0^t f(\tau)\,d\tau = q\beta \quad \text{where } \beta = \lim_{t\to\infty} x(t)$$

is the final fraction of plants which become infected. A hint is that $x(t)$ is monotonically increasing and bounded above by 1, and $f(\tau)$ consists of delayed versions of $x$.

**(iv)** Show that $\beta$ satisfies the nonlinear equation

$$\beta = 1-(1-\alpha)\exp(-Kq\beta).$$

Note that $\beta \neq 1$, except in the trivial case $\alpha = 1$: regardless of how long the epidemic continues, a positive fraction of plants remain uninfected.

*Solution.* We first show that the solution to the differential equation

$$\frac{dx}{dt} = Kf(t)(1-x) \quad \text{is} \quad x(t) = 1-(1-\alpha)\exp\left(-K\int_0^t f(\tau)\;d\tau\right). \tag{1.13}$$

We have

$$\frac{1}{1-x}\;dx = Kf(t)\;dt.$$

We integrate both sides — the right side from $\tau = 0$ to $\tau = t$. As for the left side, the limits are from $\alpha$ to $x$. So,

$$-\ln|1-x| + \ln|1-\alpha| = K\int_0^t f(\tau)\;d\tau.$$

The result follows with some algebraic manipulation.

**(i)** Use the substitution $u = \tau - a$ on the left integral.

**(ii)** By the definition of $f$, we have

$$\begin{aligned}\int_0^t f(\tau)\;d\tau &= \int_0^t x(t-p)\;d\tau - \int_0^t x(t-p-q)\;d\tau \\ &= \int_0^{t-p} x(\tau)\;d\tau - \int_0^{t-p-q} x(\tau)\;d\tau \\ &= \int_{t-p-q}^{t-p} x(\tau)\;d\tau\end{aligned}$$

Here, the second equality uses **(i)**.

**(iii)** From **(ii)**, we have

$$\lim_{t\to\infty}\int_0^t f(\tau)\;d\tau = \lim_{t\to\infty}\int_{t-p-q}^{t-p} x(\tau)\;d\tau. \tag{1.14}$$

We are now integrating $x$ over an interval of length $q$. As hinted from the blue part of the problem, $x$ denotes the fraction of plants that are infected at time $t$, so $x$ is monotonically increasing and bounded by 1. By the monotone convergence theorem, $x$ converges to 1. As such, we can approximate the integral in (1.14) by considering the area of a rectangle of width $q$ and height $\beta$, where $\beta$ was defined earlier.

**(iv)** This is trivial from (1.13). □

Chapter 2

# Models using Second-Order Differential Equations

## 2.1 Introduction

We will need to study ordinary differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x),$$

where $a,b,c$ are real constants and $f(x)$ is some given function. There is a systematic way of solving such ODEs.

Observe that since there are two derivatives present, we would need to integrate twice. Integrating once yields a constant, so the general solution of a second-order ODE must involve exactly two constants. To find them, we generally work with the initial conditions $y(0)$ and $y'(0)$ (which are usually known quantities). We then end up with two equations for two unknowns, which determine the two constants.

**Definition 2.1 (characteristic equation).** Consider the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Its characteristic equation is

$$a\lambda^2+b\lambda+c=0.$$

Note that in Definition 2.1, the characteristic equation is also known as the auxiliary equation. Note that here, we have taken $f(x)=0$. To find two solutions $S_1$ and $S_2$ to the equation, we consider the corresponding characteristic equation $a\lambda^2+b\lambda+c=0$. The idea here is that if $\lambda$ is real, then $e^{\lambda x}$ is a solution. Usually, we obtain two solutions[1], i.e.

[1] Here is a fun exercise that is related to ST2131. Given the quadratic equation $Ax^2+Bx+C=0$ with $A,B,C\sim U(0,1)$, i.e. $A,B,C$ are uniformly distributed on the interval $(0,1)$, what is the probability that the roots of the quadratic equation are real?

two numbers $\lambda_1$ and $\lambda_2$. As such, we obtain two solutions $S_1 = e^{\lambda_1 x}$ and $S_2 = e^{\lambda_2 x}$.

However, two things can potentially go wrong.

- **Case 1:** The quadratic equation might have only one root $\lambda$ (which must be real since $a, b, c$ are real). Then, we will take $S_1 = e^{\lambda x}$ and $S_2 = xe^{\lambda x}$. One should verify by direct substitution that this indeed works.
- **Case 2:** We might obtain two solutions, which are complex. In fact, by the conjugate root theorem, the roots of the quadratic equation form conjugate pairs. We focus on one of them, and write it as

  $$\lambda = \alpha + \beta i \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

  Then, we take

  $$S_1 = e^{\alpha x} \cos \beta x \quad \text{and} \quad S_2 = e^{\alpha x} \sin \beta x.$$

  Again, one should substitute these into the differential equation to be convinced that $S_1$ and $S_2$ are indeed solutions. Actually, these do not look so strange if we recall Euler's formula, which states that

  $$e^{i\theta} = \cos \theta + i \sin \theta.$$

In all cases, we will be able to deduce the solutions $S_1$ and $S_2$, provided that $a, b, c$ are constants.

For the case where $f(x) \neq 0$, suppose we have some *miraculous* way of deducing one solution to the differential equation. Call this solution $y = P(x)$, known as the particular solution. Then, we can find the general solution to the differential equation as follows. Consider

$$\begin{aligned} a\frac{d^2 S_1}{dx^2} + b\frac{dS_1}{dx} + cS_1 &= 0 \\ a\frac{d^2 S_2}{dx^2} + b\frac{dS_2}{dx} + cS_2 &= 0 \\ a\frac{d^2 P}{dx^2} + b\frac{dP}{dx} + cP &= 0 \end{aligned}$$

We multiply the first equation by an arbitrary real number $A$ and multiply the second by an arbitrary real number $B$. Thereafter, we add the three equations to obtain

$$a(AS_1 + BS_2 + P)'' + b(AS_1 + BS_2 + P)' + c(AS_1 + BS_2 + P) = f(x),$$

so we infer that $AS_1 + BS_2 + P$ is the general solution to the differential equation! This works due to the linearity of the derivative operator, and because the equation is also linear (inherently used the principle of superposition here). This method is not applicable to non-linear ODEs though.

Example 2.1. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^{2x}.$$

*Solution.* Note that the differential equation can be written as

$$\frac{d^2y}{dx^2} - y = 0 + e^{2x}.$$

We first find the complementary solution. That is, the set of all $y$ such that

$$\frac{d^2y}{dx^2} - y = 0.$$

The characteristic equation is $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$. Hence, $S_1 = e^x$ and $S_2 = e^{-x}$.

Now, we find the particular solution. The only way to obtain $e^{2x}$ on the RHS is if it is already there on the LHS. As such, we try $P(x) = ce^{2x}$, where $c$ has to be found. Since $P$ satisfies the differential equation, we have

$$4ce^{2x} - ce^{2x} = e^{2x} \quad \text{which implies} \quad c = \frac{1}{3}.$$

Hence, $P = e^{2x}/3$. Combining the complementary solution and the particular solution yields the general solution, which is

$$y = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}.$$

□

Example 2.2 (MA3264 AY25/26 Sem 1 Tutorial 3). Consider the differential equation $\ddot{y} + y = 0$, with $y(0) = 0$ and $\dot{y}(0) = 1$. Solve it using the power series method as in Example 1.4. Infer that one needs two initial conditions to obtain a specific solution to a second-order ordinary differential equation.

*Solution.* Let

$$y = a_0 + a_1 t + a_2 t^2 + \ldots.$$

Then,

$$\dot{y} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots \quad \text{so} \quad \ddot{y} = 2a_2 + 6a_3 t + \ldots$$

Actually, we can write the differential equation as

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

We need to match the exponents of $t$, so

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} a_{n-2} t^{n-2} = 0 \quad \text{so} \quad \sum_{n=2}^{\infty} \left(n(n-1)a_n + a_{n-2}\right) t^{n-2} = 0.$$

We must have

$$n(n-1)a_n + a_{n-2} = 0 \quad \text{so} \quad a_n = -\frac{a_{n-2}}{n(n-1)}.$$

We have

$$a_2 = -\frac{a_0}{2\cdot 1} \text{ and } a_4 = -\frac{a_2}{4\cdot 3} \quad \text{so} \quad a_4 = \frac{a_0}{4!}.$$

One can deduce for $n$ even,

$$a_n = (-1)^{\frac{n}{2}} \cdot \frac{a_0}{n!}$$

and for $n$ odd,

$$a_n = (-1)^{\frac{n-1}{2}} \cdot \frac{a_1}{n!}.$$

So,

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + \frac{a_1}{1!}t + \frac{a_0}{2!}t^2 - \frac{a_1}{3!}t^3 + \frac{a_0}{4!}t^4 + \ldots = a_0 \cos t + a_1 \sin t.$$

When $t=0$, $y=0$ so $a_0=0$. When $t=0$, $\dot{y}=1$. As $y=a_1\cos t$, we have $a_1=1$. Hence, $y=\cos t$. □

However, there are times where the method to finding the particular solution in Example 2.1 does not work. Let us take a look at Example 2.3.

Example 2.3. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^x.$$

Finding the complementary solution is precisely the same as Example 2.1 since both differential equations share the same characteristic equation. Now, if we try $P(x) = ce^x$ as our particular solution, we will see that $0 = e^x$, which is an obvious error. As such, we need to amend the particular solution (the method to finding the particular solution is somewhat systematic) — try $P(x) = cxe^x$. We will see that

$$\frac{d^2P}{dx^2} - P = 2ce^x \quad \text{which implies} \quad 2c = 1.$$

Hence, $c = 1/2$ and the desired general solution is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x.$$

Examples 2.1 and 2.3 are great examples of finding particular solutions. The given method works because if we have an exponential function on the RHS, taking derivatives of exponential functions would give exponential functions. Similarly, it always works for polynomials and for products of exponential functions with polynomials. However, this method does not work if we have functions like $\tan x$ on the RHS!

Example 2.4. Now, what if we wish to solve a differential equation like

$$\frac{d^2y}{dx^2} + y = \cos x?$$

The easy way to handle this is to remember that $\cos x$ and $\sin x$ are really just names of the real and imaginary parts of $e^{ix}$ respectively. As such, consider $z(x)$ to be a complex function such that $\mathrm{Re}(z) = y$. Then, say we have the equation

$$\frac{d^2z}{dx^2} + z = e^{ix}.$$

This is easy to solve because we know what to do when we have an exponential function on the RHS! As such, we solve for $z$. As we are interested in $y$, upon finding $z$, we just take the real part of that.

Again, we first find the complementary solution. We first solve

$$\frac{d^2y}{dx^2}+y=0.$$

The characteristic equation is $\lambda^2+1=0$, so $\lambda=\pm i$. Hence,

$$S_1=\cos x \quad \text{and} \quad S_2=\sin x.$$

Next, try $P(x)=ce^{ix}$, which does not work. As such, we try $P(x)=cxe^{ix}$, for which we obtain $2ice^{ix}=e^{ix}$. So,

$$c=\frac{1}{2i}=-\frac{1}{2}i.$$

Hence,

$$P(x)=-\frac{1}{2}ixe^{ix}=-\frac{x}{2}(-\sin x+i\cos x).$$

The real part of $P$ is $x\sin x/2$, so the general solution to the differential equation is

$$y=A\cos x+B\sin x+\frac{1}{2}x\sin x.$$

Example 2.5 (MA3264 AY25/26 Sem 1 Tutorial 3). Find particular solutions to the differential equation

$$\frac{d^2y}{dx^2}-y=2x\sin x.$$

*Solution.* The particular solution should be of the form

$$y_p=(Ax+B)\sin x+(Cx+D)\cos x.$$

Hence,

$$\frac{d^2y_p}{dx^2}=-(Ax+2C+B)\sin x-(Cx+D-2A)\cos x.$$

Substituting these into the differential equation, one can deduce that

$$y_p=-x\sin x-\cos x$$

is a particular solution. □

Example 2.6 (MA3264 AY25/26 Sem 1 Tutorial 1). Find particular solutions to the differential equation

$$\frac{d^2y}{dx^2}+4y=\sin^2 x.$$

*Solution.* The trick is to use the identity $\cos 2x=1-2\sin^2 x$, then make $\sin^2 x$ the subject. One can check that

$$y_p=\frac{1}{8}-\frac{1}{8}x\sin 2x$$

is a particular solution. □

Example 2.7 (MA3264 AY16/17 Sem 1 Quiz 1). Solve the resonance equation

$$\frac{d^2x}{dt^2} + x = \sin(t),$$

assuming that $x(0) = \frac{dx}{dt}(0) = 0$, and find $x(\pi/2)$.

*Solution.* We first solve the homogeneous equation

$$\frac{d^2x}{dt^2} + x = 0.$$

The characteristic equation is $m^2 + 1 = 0$ so $m = \pm i$. The solution to the homogeneous equation is

$$x_h = A\cos t + B\sin t.$$

We then find the particular solution $x_p$. Assume that

$$x_p = C\sin t + D\cos t.$$

Then, $x_p'' = -C\sin t - D\cos t$. Substituting it into the differential equation shows that $\sin t = 0$, so this particular solution does not work. Instead, try

$$x_p = Ct\sin t + Dt\cos t.$$

Then,

$$x_p'' = (-Ct - 2D)\sin t + (2C - Dt)\cos t.$$

Since

$$-2D\sin t + 2C\cos t = \sin t.$$

Hence, $C = 0$ and $D = -\frac{1}{2}$. As such, the particular solution is

$$x_p = -\frac{1}{2}t\cos t.$$

So,

$$x = A\cos t + B\sin t - \frac{1}{2}t\cos t.$$

Since $x(0) = 0$, then $A = 0$. One can then show that $B = \frac{1}{2}$, so

$$x(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

Hence, $x\left(\frac{\pi}{2}\right) = \frac{1}{2}$. □

## 2.2 Stability

Definition 2.2 (pendulum equation). Consider a pendulum. Let $\theta$ be the angle with the vertical and let $L$ be the length of the pendulum. Then, using Physics (briefly see

Figure 2.1), one can deduce that a differential equation governing $\theta$ is as follows:

$$\frac{d^2\theta}{dt^2}+\frac{g}{L}\sin\theta=0.$$

Sometimes, $d^2\theta/dt^2$ is written as $\ddot{\theta}$, which also denotes the second derivative of the angle with respect to time.

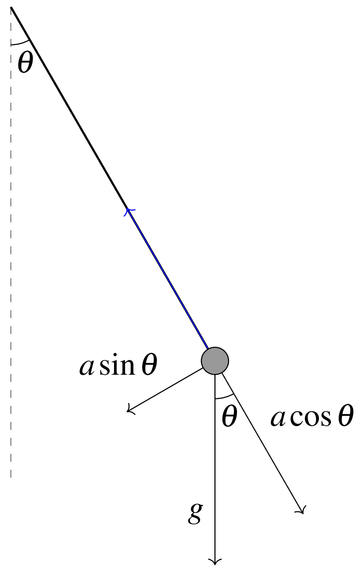



Figure 2.1: A free-body diagram of a pendulum bob

It is possible to solve the pendulum differential equation (Definition 2.2), but this involves complicated concepts in Mathematics like elliptic integrals.

Let us simplify the setup. An obvious solution is $\theta=0$, which is known as an equilibrium solution, meaning that $\theta$ is a constant function. This means that if we set $\theta=0$ initially, then $\theta$ will remain at 0 and the pendulum will not move — which of course know is correct. There is another equilibrium solution which is $\theta=\pi$. Again, in theory, if we set the pendulum exactly at $\theta=\pi$, then it will remain in that position forever. In reality, it will not due to gravity! As such, the equilibrium at $\theta=\pi$ is very much different from the one at $\theta=0$ (an important distinction).

Definition 2.3 (equilibrium). The equilibrium of an object is said to be stable if a small push away from equilibrium remains small. If the small push tends to grow large, then the equilibrium is unstable.

The concept of equilibrium is particularly important for engineers as they want vibrations of structures, engines, etc. to remain small.

We shall analyse the case where $\theta=\pi$. By Taylor's theorem, near $\theta=\pi$, we have

$$f(\theta)=f(\pi)+f'(\pi)(\theta-\pi)+\frac{1}{2}f''(\pi)(\theta-\pi)^2+\ldots,$$

and upon letting $f(\theta) = \sin\theta$, we obtain the following series expansion:

$$\sin\theta = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 + \dots$$

For small deviations away from $\pi$, note that $\theta - \pi$ is small, and $(\theta - \pi)^3$ is much smaller. So, we have the following approximation:

$$\sin\theta \approx -(\theta - \pi)$$

As such, the pendulum differential equation in Definition 2.2 can be approximated as follows:

$$\frac{d^2\theta}{dt^2} \approx \frac{g}{L}(\theta - \pi).$$

Using the substitution $\phi = \theta - \pi$, the differential equation can be written as

$$\frac{d^2\phi}{dt^2} = \frac{g}{L}\phi.$$

This equation has the general solution

$$\phi = Ae^{\sqrt{g/Lt}} + Be^{-\sqrt{g/Lt}}$$

so

$$\theta = Ae^{\sqrt{g/Lt}} + Be^{-\sqrt{g/Lt}} + \pi.$$

Since the exponential function grows very quickly, even if $\theta$ is close to $\pi$ initially, it will not stay near it very long. Very soon, $\theta$ will arrive at either $\theta = 0$ or $\theta = 2\pi$, which is far away from $\theta = \pi$. This equilibrium is unstable! So, we ask how long would it take for things to get out of control? This is determined by the quantity in the exponent of the exponential term which is $\sqrt{g/L}$. Note that it takes longer for the pendulum to fall if $L$ is large.

## 2.3 Damped Oscillations

When an object moves fairly slowly through air, the resistance due to friction is approximately proportional to its speed, and of course in the opposite direction. One would recall Hooke's law from H2 Physics. In fact, we can extend it to the following differential equation (Definition 2.4):

**Definition 2.4 (simple harmonic oscillator).**

$$m\frac{d^2x}{dt^2} + kx = 0$$

This equation describes the oscillation of a block of mass $m$ on one of a spring and a nail on the other end. Here, $x$ measures how much the spring is stretched and $k$ is a positive constant that measures the stiffness of the spring (known as the spring constant).

If we include friction which is proportional to the speed, we obtain

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0,$$

where $b$ is a positive constant known as the damping coefficient. It quantifies the resistance to motion provided by the medium (such as air or fluid), often associated with dissipative forces like friction or drag. As such, the corresponding characteristic equation is

$$m\lambda^2+b\lambda+k=0.$$

Note that $m,b,k>0$. Let us discuss the solutions to this differential equation. We consider three cases on the nature of the roots.

- **Case 1:** $\lambda_1$ and $\lambda_2$ are real, which results in overdamping

  Example 2.8. Consider the differential equation

  $$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=0.$$

  Its characteristic equation is $\lambda^2+3\lambda+2=0$, which yields the roots $\lambda=-1$ and $\lambda=-2$. The general solution is

  $$x=B_1e^{-t}+B_2e^{-2t}.$$

  We see that the motion rapidly dies away to zero, which implies that there is much friction.

- **Case 2:** $\lambda_1$ and $\lambda_2$ are complex, which results in underdamping

  Example 2.9. Consider the differential equation

  $$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+13x=0.$$

  Its characteristic equation is $\lambda^2+4\lambda+13=0$, which yields the roots $\lambda=-2\pm3i$. The general solution is

  $$x=B_1e^{-2t}\cos 3t+B_2e^{-2t}\sin 3t,$$

  which by the $R$-formula (from O-Level Additional Mathematics), can be also written as

  $$x=\sqrt{B_1^2+B_2^2}e^{-2t}\cos\left(3t-\frac{\pi}{4}\right).$$

  This acts like a simple harmonic oscillator, where the amplitude $\sqrt{B_1^2+B_2^2}e^{-2t}$ is a function of time. Note that in this problem, there are two independent time scales. First, the factor $e^{-2t}$ determines how quickly the oscillations decay over time. This decay is governed by the real part of the roots. Next, the angular frequency of oscillation is determined by the imaginary part of the roots. The oscillation period is

  $$T=\frac{2\pi}{\omega}=\frac{2\pi}{3}\quad\text{where } \omega=3 \text{ is the angular frequency.}$$

  This represents the rapidity of oscillations within the decaying envelope.

## 2.4 Forced Oscillations

Now, consider the case where an external motor is attached to the block of mass $m$. This motor exerts a force of $F_0 \cos \alpha t$, where $F_0$ is the amplitude of the external force and $\alpha$ is the frequency. If $F_0 = 0$, then by Newton's second law, we have

$$m\frac{d^2x}{dt^2} + kx = 0$$

so we obtain the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}. \tag{2.1}$$

Here, $\omega$ is the frequency that the system has if we leave it alone, i.e. its natural frequency. It is not related to $\alpha$.

If $F_0 \neq 0$, then we have

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \alpha t.$$

Let $z$ be a complex function that satisfies the differential equation

$$m\frac{d^2z}{dt^2} + kz = F_0 e^{i\alpha t}.$$

The real part, $\operatorname{Re} z$, satisfies this differential equation, so we can solve for $z$ and then take the real part. Try $z = ce^{i\alpha t}$ to be a solution. One can deduce that

$$c = \frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad \operatorname{Re} z = \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t.$$

We conclude that the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t,$$

where $\delta$ is some constant (we will explain in just a bit). Note that upon differentiation, we obtain

$$\frac{dx}{dt} = -A\omega \sin(\omega t - \delta) - \frac{aF_0/m}{\omega^2 - \alpha^2} \sin \alpha t.$$

The constants $A$ and $\delta$ are fixed. One can deduce these values from $x(0)$ and $\dot{x}(0)$ as usual, where we recall that $\dot{x}(0)$ is $dx/dt$ evaluated at $t = 0$.

Example 2.10. As an example, if $x(0) = \dot{x}(0) = 0$, then we have

$$A\cos\delta + \frac{F_0/m}{\omega^2 - \alpha^2} = 0 \quad \text{and} \quad A\omega \sin\delta = 0 \quad \text{respectively.}$$

Assuming that $F_0 \neq 0$, we cannot have $A = 0$, which forces $\delta = 0$. Hence,

$$A = -\frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad x = \frac{F_0/m}{\omega^2 - \alpha^2}(\cos \alpha t - \cos \omega t).$$

Notice that the amplitude function

$$A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left(\frac{\alpha - \omega}{2} t\right)$$

has a maximum value

$$\left| \frac{2F_0/m}{\alpha^2 - \omega^2} \right|$$

which becomes very large when $\alpha$ is very close to $\omega$. What happens if we let $\alpha \to \omega$? We have

$$A(t) = \frac{2F_0/m}{\alpha + \omega} \cdot \frac{\sin\left(\frac{\alpha-\omega}{2} t\right)}{\alpha - \omega} \longrightarrow \frac{F_0}{m\omega} \cdot \frac{t}{2} = \frac{F_0 t}{2m\omega}$$

by L'Hopital's rule. So in this limit, we have

$$x = \frac{F_0 t}{2m\omega} \sin(\omega t)$$

and we see that the oscillations go completely out of control. This situation is called resonance. We see that if a system is forced in a way that agrees with its own natural frequency, it can oscillate uncontrollably. A vast number of things can be modelled using the concept of resonance. For example, giant tides (Figure 2.2). This can be very dangerous! However, in reality resonance does not get completely out of control, because we cannot really ignore friction (or *resistance* in the case of an electrical circuit).

Figure 2.2: The Seven Sisters in England

So we should really solve

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\alpha t)$$

or

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\alpha t}.$$

Set $z = ce^{i\alpha t}$ and take the real part at the end. One can solve for $c$. Similarly,

$$\frac{F_0[k - m\alpha^2 - ib\alpha]}{(k - m\alpha^2)^2 + b^2\alpha^2} \times [\cos(\alpha t) + i\sin(\alpha t)]$$

has real part

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}.$$

To this we should add the general solution to $m\ddot{x} + b\dot{x} + kx = 0$. However, we already know that looks like — whether overdamped or underdamped, the solution rapidly (exponentially) tends to zero. We call it the *transient*. So after the transient dies off, we are left with this expression. Recall that any expression of the form $C\cos x + D\sin x$ can be written as

$$C\cos x + D\sin x = \sqrt{C^2 + D^2}\cos(x - \gamma) \quad \text{where } \tan\gamma = \frac{D}{C}.$$

So, here we have

$$x(t) = \frac{1}{m}\frac{F_0\cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

As such, the system eventually settles down into a steady oscillation, but at a frequency $\alpha$ and not $\omega$. This means we can steer the system away from its natural frequency towards any other frequency we want. Also, the *amplitude* of this oscillation is a function of $\alpha$. That is,

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

The graph of this is called the amplitude response curve. The amplitude (usually) has a maximum at a certain value of $\alpha$, found by minimising

$$(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2.$$

If we are at this maximum, we say we are at resonance in this case. Notice that if $b = 0$, then this function is minimised by choosing $\alpha \to \omega$, which implies $A(\alpha) \to \infty$, taking us back to the discussion of resonance in the zero–friction case. This is why we call this resonance in the frictional case.

Example 2.11 (MA3264 AY25/26 Sem 1 Tutorial 3). Consider a forced damped harmonic oscillator, which is modelled by the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

as in (2.1). It is known that the amplitude response function $A$ is a function of $\alpha$, the input frequency. That is,

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

Find the maximum of this function. Note that there are different answers when the frictional constant b is large and when it is small. Show that, when $b$ is so small that the dimensionless quantity $b^2/m^2\omega^2$ can be neglected, the maximal amplitude is proportional to $1/b$.

*Solution.* Differentiating $A$ with respect to $\alpha$ and setting the derivative equal to zero, we get

$$4\alpha\left(\alpha^2-\omega^2\right)+\frac{2b^2\alpha}{m^2}=0 \quad \text{so} \quad \alpha^2=\omega^2-\frac{b^2}{2m^2}.$$

Of course, the left side cannot be negative, so if $b \geq \sqrt{2}m\omega$, then there is no resonance; this is the situation described above. In that case, the maximum amplitude is at $\alpha=0$ and is given by $\frac{F_0}{m\omega^2}$. Otherwise, the maximal value of the amplitude is obtained by substituting this value of $\alpha$ into $A(\alpha)$. With some algebraic manipulation, we have

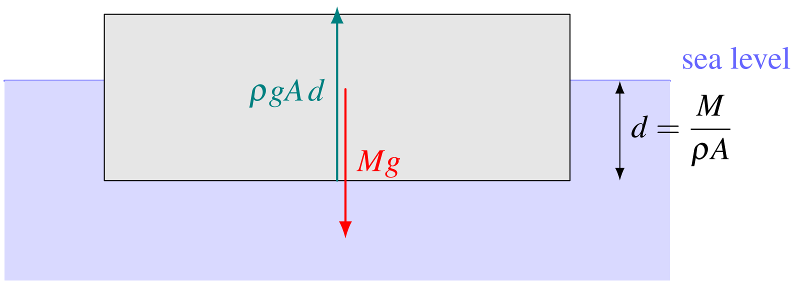
$$A_{\text{resonance}}=\frac{F_0/b\omega}{\sqrt{1-\left(b^2/4m^2\omega^2\right)}}.$$

If $\frac{b^2}{m^2\omega^2}$ is negligible, then this is approximately $\frac{F_0}{b\omega}$. That is, the resonance amplitude grows without limit as $b$ becomes smaller. □

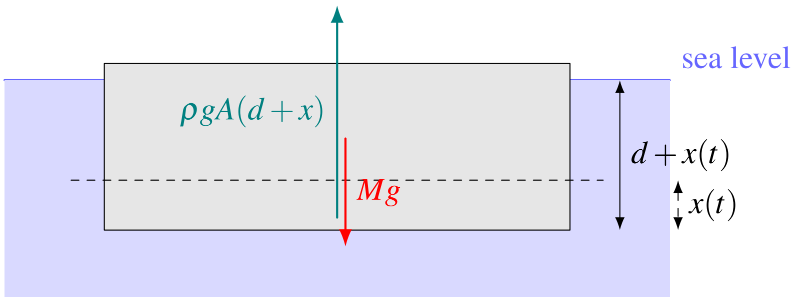
Example 2.12 (MA3264 AY25/26 Sem 1 Tutorial 4). A fully loaded large oil tanker can be modelled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, equal to $A$. Archimedes' principle states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let $\rho$ be the mass density of seawater, and let $M$ be the mass of the ship, so that its weight is $Mg$, where $g$ is 9.8 m/s$^2$. When the ship is at rest, find the distance $d$ from sea level to the bottom of the ship. This is called the *draught* of the ship.

Suppose now that the ship is not at rest; instead it is moving in the vertical direction. Let $d+x(t)$ be the distance from sea level to the bottom of the ship, where $d$ is the draught as above. Show that, if gravity and buoyancy are the only forces acting on the ship, it will bob up and down with an angular frequency given by $\omega = \sqrt{\rho A g/M}$.

**(a) Equilibrium**




**(b) Displaced**




Next, suppose that waves from a storm strike the ship, which is initially at rest with $x(0) = 0$, and exert a vertical force $F_0 \cos(\omega t)$ on the ship, where $F_0$ is the amplitude of the wave force. Let $H$ be the height of the deck of the ship above sea level when the ship is at rest. We assume that the ship is heavily loaded, so $H$ is much less than $d$. Write down a formula which allows you to compute when the ship sinks. That is, find an equation satisfied by $t_{\text{sink}}$, the time at which the ship's deck first goes under water. You do not need to solve this equation.

*Solution.* When the ship is at rest, the part of it which is under sea level has a volume of $Ad$, where $A$ denotes the cross-sectional area and $d$ denotes the distance from sea level to the bottom of the ship. This is the volume of seawater that has been displaced by the ship. Suppose the density of the seawater is $\rho$. Then, the mass of water that has been displaced by the ship is $\rho \cdot Ad = \rho Ad$. So, the weight is $\rho Adg$. This upward force balances the weight of the ship, so

$$\rho Adg = Mg \quad \text{which implies} \quad d = \frac{M}{\rho A}.$$

Now, if the ship is moving and the distance from sea level to the bottom of the ship is $d + x$, where $x$ is a function of time. Recall from Newton's second law that force is the product of mass and acceleration. Taking the downwards direction to be positive, the buoyancy force is

$$-\rho A(d+x)g.$$

So, we have

$$M\ddot{x} = Mg - \rho A(d+x)g.$$

Using our formula for $d$, we have

$$\ddot{x} = -\frac{\rho A g}{M} x.$$

This represents simple harmonic motion with angular frequency $\omega = \sqrt{\rho A g / M}$ as claimed. The ship will bob up and down at this frequency. Note the inverse dependence on $M$, which is to be expected, but also that the frequency increases if $A$ is large, which is not so obvious.

Lastly, suppose waves from a storm strike the ship. Taking into account the force acting on the waves and taking downwards to be negative, we have

$$M\ddot{x} = Mg - \rho A (d + x) g + F_0 \cos(\omega t).$$

Recall that at equilibrium (i.e. $x = 0$), we have $Mg = \rho A g d$. Substituting $Mg = \rho A g d$ into the equation of motion simplifies it to

$$M\ddot{x} + \rho A g x = F_0 \cos(\omega t).$$

At rest, the deck is at height $H$ above sea level. If the bottom of the ship is at depth $d + x$, then the deck (which is a height $H$ above that bottom) is located above sea level by the amount $H - x$. The deck is above water so long as $H - x > 0$. The instant the deck just touches the water surface is when $H - x = 0$, i.e. $x = H$. So, the sinking time $t_{\text{sink}}$ is the first time for which this happens. Let $x(t)$ be the solution to

$$M\ddot{x} + \rho A g x = F_0 \cos(\omega t)$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Then the time at which the deck first goes underwater is defined implicitly by $x(t_{\text{sink}}) = H$. □

## 2.5 The Phase Plane Method

The phase plane method is a geometrical way of analysing second-order differential equations by turning them into first-order systems and studying their behaviour graphically — not as functions of time, but as trajectories in a plane whose axes are the variable and its derivative. Recall Definition 2.4 on the differential equation governing simple harmonic oscillation, which is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

As such, we can rewrite it as follows:

$$m\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = -kx.$$

Integrating both sides yields

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = -\frac{1}{2}kx^2 + E,$$

where $E$ is a constant. In fact this is not surprising as $E$ is the total energy of the system! Since $dx/dt$ denotes velocity, one would know that

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \text{ denotes the kinetic energy} \quad \text{and} \quad \frac{1}{2}kx^2 \text{ denotes the potential energy}$$

of the oscillator respectively. One should recall that the fact that $E$ is constant is known as the conservation of energy.

This idea of turning time derivatives into space derivatives can be very useful when studying certain kinds of second-order non-linear differential equations. For example, we recall the pendulum problem (Definition 2.2) which is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \quad \text{with initial conditions} \quad \theta(0) = 1 \text{ and } \dot{\theta}(0) = 1.$$

Here, we have taken $g/L = 1$. Although we cannot find elementary solutions for this differential equation (recall that we can do so but the solution involving elliptic integrals would be non-elementary), we can still can gain some insights such as determining the maximum value of $\theta$. This is simple as we have $\dot{\theta}(t) = 0$. Solving yields $\theta_{\max} \approx 1.53$.

In fact, there is a nice way of thinking about what we did. One can look at the equation involving $\dot{\theta}$ and use it to think of $\dot{\theta}$ as a function of $\theta$. If we graph that function, we can see that the graph is a closed curve. As time goes by, the point $(\theta, \dot{\theta})$ moves around and around the closed curve. As such, the solution must be a periodic function of time. This makes sense as the physical system is a pendulum. We call this the phase plane method.

Example 2.13. We analyse the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{2}\cos y = 0 \quad \text{with initial conditions} \quad y(0) = 0 \text{ and } \dot{y}(0) = 1.$$

Note that the equation describes a non-linear oscillator, which should still typically produce bounded and oscillatory motion. In fact, for large $t$, $y(t)$ tends to infinity! What is really happening here? We note that we can write the differential equation as

$$\begin{aligned}
\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2\right] + \frac{d}{dy}\left(\frac{1}{2}\sin y\right)\frac{dy}{dt} &= 0 \\
\frac{d}{dt}\left[\left(\frac{dy}{dt}\right)^2 + \sin y\right] &= 0 \\
\left(\frac{dy}{dt}\right)^2 + \sin y &= c
\end{aligned}$$

Substituting the initial conditions yields $c = 1$, so

$$\left(\frac{dy}{dt}\right)^2 + \sin y = 1,$$

which is the phase plane equation for the differential equation. On the $(y, \dot{y})$ phase plane, as the system moves from the point $(0,1)$ to the point $(\pi/2, 0)$, it actually never gets there! One can use the method of separation of variables to obtain

$$t = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin y}}\, dy = \int_0^{\pi/2} \frac{\sqrt{1+\sin y}}{\cos y}\, dy \geq \int_0^{\pi/2} \sec y\, dy$$

which is infinite. As such, the *correct* graph is not the one produced by Wolfram Mathematica (for instance), but rather the one in which $y$ asymptotically approaches $\pi/2$. As such, the phase plane method helps us spot this error made by computers.

Example 2.14 (MA3264 AY25/26 Sem 1 Tutorial 4). Use the phase plane method to find the largest and smallest possible values of $x(t)$ if $x(0) = 0$, $\dot{x}(0) = 1$, and $x(t)$ satisfies $\ddot{x} = \cos x$.

*Solution.* We have

$$\frac{d^2x}{dt^2} - \cos x = 0.$$

Let $v = \dot{x}$. Recall that

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = \frac{d^2x}{dt^2}.$$

Hence,

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \sin x\right) = 0 \quad \text{so} \quad \frac{1}{2}v^2 - \sin x = c.$$

Since $x(0) = 0$ and $v(0) = 1$, then $c = \frac{1}{2}$. As such, $v^2 = 2\sin x + 1$. Since $v \geq 0$, then $2\sin x + 1 \geq 0$. The turning points occur at $v = 0$, so we must have $2\sin x + 1 = 0$. One can then easily find the largest and smallest possible values of $x$. $\square$

Example 2.15 (MA3264 AY25/26 Sem 1 Tutorial 5). Consider the differential equation

$$\frac{d^2y}{dx^2} + y = \frac{1}{2}\cosh y,$$

with $y(0) = 0$, $y'(0) = \sqrt{0.3}$. Show that the equivalent first–order equation is

$$\left(\frac{dy}{dx}\right)^2 + y^2 = \sinh y + 0.3,$$

and hence find the maximum value of $y$.

*Solution.* We have

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] + \frac{1}{2}\frac{d}{dx}\left(y^2\right) = \frac{1}{2}\frac{d}{dx}(\sinh y)$$

so

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y^2 = \sinh y + c.$$

When $x = 0$, $y = 0$ and $y' = \sqrt{0.3}$. One can deduce that $c = 0.3$ so the equivalent first-order equation is

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y^2 = \sinh y + 0.3.$$

To find the maximum value of $y$, we set the derivative to be zero, so $y^2 = \sinh y + 0.3$. One can use a computer to find the maximum value of $y$. $\square$

**Example 2.16 (MA3264 AY16/17 Sem 1 Quiz 1).** The function $y(x)$ satisfies the equation

$$\frac{d^2y}{dx^2} = \sin(y) + \cos(y),$$

as well as $y(0) = y'(0) = 0$. Find $\frac{dy}{dx}$ when $y = \pi/4$. Find the maximum value of $y(x)$.

*Solution.* The trick is to multiply both sides of the differential equation by $y'$ to obtain

$$y'y'' = (\sin y + \cos y)\, y'.$$

Note that the derivative of $\frac{1}{2}(y')^2$ is $y'y''$ so we shall consider the differential equation

$$\frac{d}{dx}\left(\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right) = (\sin y + \cos y)\frac{dy}{dx}.$$

Also, observe that

$$\frac{d}{dx}(\sin y - \cos y) = (\cos y + \sin y)\frac{dy}{dx}.$$

Hence, the differential equation becomes

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2 = \sin y - \cos y + c.$$

Since $y'(0) = 0$, then $c = 1$. So,

$$\left(\frac{dy}{dx}\right)^2 = 2\sin y - 2\cos y + 2.$$

Since $y(0) = y'(0) = 1$, then taking the positive square root yields $y'\left(\frac{\pi}{4}\right) = \sqrt{2}$. We then find the maximum value of $y(x)$. Set $y' = 0$ so $2\sin y - 2\cos y + 2 = 0$. As such, $\sin y - \cos y = -1$, so $\sqrt{2}\sin\left(y - \frac{\pi}{4}\right) = -1$. Hence, the maximum value of $y$ is $3\pi/2$. $\square$

Chapter 3

# Population Models

## 3.1 The Malthusian Model for Population Growth

Population modelling is a crucial area of applied Mathematics that uses differential equations to understand the dynamics of populations. These models can reveal surprising and sometimes counterintuitive behaviors in various species, from fish to humans.

The total population of a country, denoted as $N(t)$, is clearly a function of time. For simplicity, $N$ can be measured in millions, meaning values less than 1 are still meaningful. Given the current population, can we predict how it will change? To begin, consider the per capita birth rate, $B$, which represents the number of babies born per second, divided by the total population at that moment. The value of $B$ varies — it might be small in a populous country or large in a smaller one, depending on societal factors like cultural attitudes toward marriage and children. $B$ could depend on time $t$ and the current population $N$.

For simplicity, we assume that $B$ is constant, i.e. people will always have as many children as possible, regardless of time or population size. In this case, the number of births over a small time interval $\delta t$ is given by $BN\delta t$.

Similarly, consider the per capita death rate $D$, which also depends on $t$ (i.e. better healthcare) or $N$ (i.e. overcrowding). Assuming $D$ is constant, the number of deaths over $\delta t$ is $DN\delta t$.

Assuming no immigration or emigration, the change in population, $\delta N$, over $\delta t$ is simply the difference between births and deaths. That is,

$$\delta N = \text{births} - \text{deaths} = (B - D) N \delta t.$$

Recall from MA2002 that we can divide throughout by $\delta t$ and take the limit as $\delta t \to 0$. We then obtain the differential equation

$$\frac{dN}{dt} = (B-D)N = kN \quad \text{where } k \text{ is the net growth rate.}$$

This simple model was first proposed by Thomas Malthus in 1798, laying the foundation for what is now known as Malthusian population growth (Definition 3.1).

**Definition 3.1 (Malthusian growth model).** Let $N$ denote the current population, $B$ and $D$ denote the birth rate and death rate respectively. Then,

$$\frac{dN}{dt} = (B-D)N.$$

The Malthusian model predicts exponential growth if $k > 0$ or exponential decay if $k < 0$, assuming constant birth and death rates. To see why, one can easily solve the differential equation in Definition 3.1 to deduce that

$$N = N_0 e^{kt} \quad \text{where } k = B - D.$$

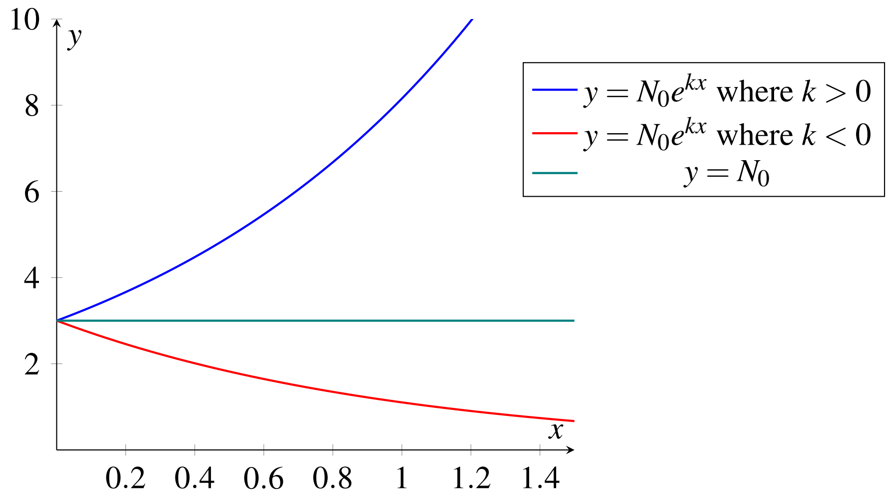



Figure 3.1: Interpretation of the Malthusian growth model

Malthus' model is interesting as it shows that static behaviour patterns can lead to disaster. As $e^{kt}$ grows so quickly for $k > 0$, Malthus' assumptions must eventually go wrong — obviously there is a limit to the possible population. Eventually, if we do not control $B$, $D$ will have to increase, so we have to assume that $D$ is a function of $N$. Hence, we turn to Verhulst's model, which will be discussed in Chapter 3.2.

**Example 3.1 (MA3264 AY25/26 Sem 1 Tutorial 5).** The bacteria in a certain culture number 10000 initially. Two and a half hours later, there are 11000 of them. Assuming a Malthus model, how many bacteria will there be 10 hours after the start of the experiment? How long will it take for the number to reach 20000?

*Solution.* Assume that the rate of growth of the population is proportional to the population. That is,

$$\frac{dN}{dt} = kN.$$

Solving the differential equation yields $N = Ce^{kt}$. Since $N(0) = 10000$ and $N(2.5) = 11000$, we have $C = 10000$ and $k = 0.4\ln 1.1$. Hence,

$$N = 10000e^{0.4\ln 1.1t}.$$

We leave the remaining computations as an exercise. □

Example 3.2 (MA3264 AY16/17 Sem 1 Quiz 1). A population of laboratory rats has a birth rate per capita given by $2Ke^{-\frac{t}{T}}$, where $K = 2$/year and $T = 0.5$ years. The death rate per capita is $Ke^{-\frac{t}{T}}$. If we start with 1000 rats, how many will we have in the (very) long run?

*Solution.* This represents a Malthusian growth model. We have

$$\frac{dP}{dt} = 2Ke^{-\frac{t}{T}}P - Ke^{-\frac{t}{T}}P = KPe^{-\frac{t}{T}}.$$

As mentioned, $K = 2$ and $T = 0.5$. So,

$$\frac{dP}{dt} = 2Pe^{-2t}.$$

Integrate both sides and substitute $N(0) = 1000$. Then, show that the limiting value is $1000e$. □

Example 3.3 (MA3264 AY25/26 Sem 1 Tutorial 5). On the island of *Orpsengia*, the human birth and death rates per capita are constant, and the population of the island has been doubling every 20 years. However, one day, several pirate ships arrive. All of the island women under the age of 50 decide to elope with the glamorous pirates, taking their children with them. After that, the remaining population of Orpsengia declines by half over the next ten years. What was the original birth rate per capita on Orpsengia? You will have to make several simplifying assumptions to solve this problem; that is ok as long as you list your assumptions carefully!

*Solution.* Suppose the original birth rate and death rate are $B$ and $D$ respectively. By assuming a Malthusian population, we have

$$\frac{dN}{dt} = (B-D)N \quad \text{so} \quad N = N_0e^{(B-D)t}.$$

As the population doubles in 20 years, when $t = 20$, we have $N = 2N_0$, so $2N_0 = N_0e^{20(B-D)}$. Hence,

$$B - D = \frac{\ln 2}{20}.$$

When the women depart, the new population model admits the differential equation

$$\frac{dN}{dt} = \left(B' - D\right)N.$$

It has solution $N = N_0 e^{B'-Dt}$. As the population declines by half over the next 10 years, when $t = 10$, $N = N_0/2$ so $10(B'-D) = \ln 2$. Hence, $B' = 0.104$. Hence, the original birth rate per capita is about 10.4%.

The assumptions made are as follows. Men and old women have same death rate as young women, which is not true in reality because men smoke, get into fights etc while on the other hand women naturally have longer life spans; the death rate of the remaining population is not changed by the departure of the girls. □

## 3.2 Verhulst's Model of Population Growth

Previously, we mentioned that the death rate $D$, should depend on $N$. A natural starting point is the simplest possible choice, i.e.

$$D = sN \quad \text{where } s \text{ is a constant.}$$

This assumption is often referred to as the logistic assumption. It captures the idea that finite resources in the environment lead to higher death rates as the population increases due to factors like starvation and disease. Hence, we obtain Verhulst's logistic growth model, which was proposed by Pierre-François Verhulst in the 19th century.

**Definition 3.2 (Verhulst's logistic growth model).** Again, let $N$ denote the current population, $B$ and $D$ denote the birth rate and death rate respectively. Then, we can write

$$\frac{dN}{dt} = BN - DN = BN - sN^2 = BN\left(1 - \frac{sN}{B}\right).$$

We shall analyse Verhulst's growth model. Suppose the initial population $N_0$ is small. Then, $N(t)$ will remain small as well. Since $N^2$ becomes negligible compared to $N$, the equation simplifies to

$$\frac{dN}{dt} \approx BN \quad \text{which has solution} \quad N = N_0 e^{Bt}.$$

Thus, for small populations, the growth is approximately exponential, as predicted by Malthus.

As the population grows, the quadratic term $sN^2$ dominates as $N^2$ increases much faster than $N$. At some point, the terms $BN$ and $sN^2$ balance, i.e. $BN = sN^2$. This happens when

$$N \approx \frac{B}{s}.$$

At this population size, the growth rate $dN/dt$ becomes zero, indicating that the population stabilises. As such, the quantity $B/s$ would be of interest.

**Definition 3.3 (carrying capacity).** In Verhulst's growth model, the value $B/s$ is called the carrying capacity of the environment, representing the maximum sustainable population under the given conditions.

Note that Verhulst's equation can be easily solved by partial fraction decomposition (see Figure 3.2 for the graph of the logistic curve). Here, we consider the possibility that we begin with a small population, i.e. $N_0 < B/s$[1].

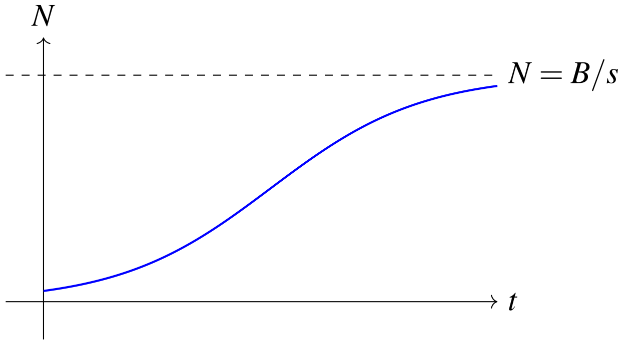



Figure 3.2: Graph of the logistic function

**Example 3.4 (MA3264 AY25/26 Sem 1 Tutorial 5).** You have 200 bugs in a bottle. Every day you supply them with food and count them. After two days you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day. Assuming that the population is given by a logistic model, find the number of bugs after 3 days. Predict how many bugs you will have eventually.

*Solution.* Assume that

$$\frac{dN}{dt} = BN\left(1 - \frac{sN}{B}\right)$$

where $s$ is a constant. Let $B = 1.5$, so

$$\frac{dN}{dt} = 1.5N\left(1 - \frac{sN}{1.5}\right).$$

Solving the differential equation yields

$$N(t) = \frac{1.5/s}{1 + \left(\frac{1.5}{sN_0} - 1\right)e^{-1.5t}}.$$

When $t = 0$, $N = 200$ and when $t = 2$, $N = 360$. Substituting these yields

$$N(t) = \frac{375}{1 + 0.875e^{-1.5t}}.$$

Then, find $N(3)$ and the limit of $N$ as $t$ approaches infinity. We omit the calculations because they are too simple. □

[1] The case where $N_0 > B/s$ will not be discussed. In this other scenario, we assume that we begin with a large population, i.e. $N_0 > B/s$. Then, the solution is monotonically decreasing, but again, the asymptotic value is the same.

## 3.3 Harvesting

One major application of mathematical modelling is in dealing with populations of animals. We wish to know how many we can eat (say fish). We will build on Verhulst's model, i.e. assume that the fish population would follow that model if we did not catch any. Next, we assume that we catch $H$ fish per unit time (say year). Then, the new differential equation representing a basic harvesting model can be written as

$$\frac{dN}{dt} = bN - sN^2 - H. \tag{3.1}$$

Again, one can use partial fraction decomposition to determine the solution to the differential equation.

Example 3.5. Recall Example 3.4, but now we assume that you are keeping the bugs not as a hobby, but because you are developing a new insecticide. Suppose that you remove 80 bugs per day from the bottle, and that all of these bugs die a painful but well-deserved death as a result of being sprayed with this insecticide.

**(a)** What is the limiting population in this case?

**(b)** What is the maximum number of bugs you can put to death per day without causing the population to die out?

*Solution.*

**(a)** Recall that the birth rate for this kind of bug is 150% per day. By considering the harvesting equation (3.1), where $D = sN$ and $s \approx 0.00400$ we have

$$\frac{dN}{dt} = 1.5N - 0.004N^2 - 80.$$

The solution to the differential equation is

$$N(t) = \frac{310.61 - 64.39Ae^{-0.986t}}{1 - Ae^{-0.986t}} \quad \text{where} \quad A = \frac{N_0 - 310.61}{N_0 - 64.39}.$$

Recall that $N_0 = 200$ from Example 3.4. Hence, the limiting population is about 311.

**(b)** The harvesting equation is now

$$\frac{dN}{dt} = 1.5N - 0.004N^2 - H.$$

We regard the right as a quadratic polynomial in terms of $N$, which has discriminant $2.25 - 0.016H$. We must have this to be $\geq 0$. If the discriminant was negative, then there are no real roots so no equilibria at all, meaning the population growth rate would always stay strictly positive or strictly negative regardless of how large or small $N$ is. This is unrealistic biologically. Hence, to have realistic population dynamics with biologically meaningful equilibrium states, we require the discriminant to be $\geq 0$, so the maximum value of $H$ is about 141. □

Example 3.6 (MA3264 AY25/26 Sem 1 Tutorial 6). Suppose that Peruvian fishermen take a fixed number of anchovies per year from an anchovy stock which would otherwise behave logistically, apart from occasional natural disasters. Say any fishing rate that is $\geq B^2/4s$, where $D = sN$, will be disastrous. Call this number $E^*$.

The fishermen want to take as many anchovies as they safely can, meaning that they want the fish to be able to bounce back from a natural disaster that pushes their population down by 10%. Advise them. That is, tell them the maximum number of fish they can take, expressed as a percentage of $E^*$. A hint is to assume that you start with the stable equilibrium population $\beta_2$, and compute the value of $E$, the harvesting rate, such that $\beta_1$, the unstable equilibrium population, becomes 90% of $\beta_2$.

*Solution.* We must have $\beta_1 = 0.9\beta_2$. The differential equation modelling the above-mentioned setup is

$$\frac{dN}{dt} = BN - sN^2 - E,$$

where $E$ denotes the harvesting rate. We can write $BN - sN^2 - E$ as $-\left(sN^2 - BN + E\right)$, which is a quadratic polynomial with roots

$$\beta_1 = \frac{B - \sqrt{B^2 - 4sE}}{2s} \quad \text{and} \quad \beta_2 = \frac{B + \sqrt{B^2 - 4sE}}{2s}.$$

So,

$$\frac{\beta_1}{\beta_2} = \frac{B - \sqrt{B^2 - 4sE}}{B + \sqrt{B^2 - 4sE}} = 0.9.$$

Making $E$ the subject of the equation yields

$$E = \frac{360}{361} \cdot \frac{B^2}{4s}$$

So, the maximum safe constant is $360/361$ times of $E^*$. □

Example 3.7 (MA3264 AY25/26 Sem 1 Tutorial 5). The sandhill crane is a beautiful Canadian bird (Figure 3.3) with an unfortunate liking for farm crops.

Figure 3.3: Sandhill cranes

For many years the cranes were protected by law, and eventually they settled down to a logistic equilibrium population of 194,600 with birth rate per capita 9.866% per year. Eventually the patience of the farmers was exhausted and they managed to have the hunting ban lifted. The farmers happily shot 10000 cranes per year, which they argued was reasonable enough since it only represents about 5% of the original population.

**(a)** Show that the sandhill crane is doomed.

**(b)** How long will it take, from the legalisation of hunting, to exterminate them?

*Solution.*

**(a)** We have $B = 0.09866$. The unharvested model admits the differential equation

$$\frac{dN}{dt} = 0.09866N\left(1 - \frac{N}{194,600}\right).$$

One hunting is legalised, the harvested model admits the differential equation

$$\frac{dN}{dt} = 0.09866N\left(1 - \frac{N}{194,600}\right) - 10000.$$

We regard the quadratic polynomial on the right as a function in terms of $N$, which has negative discriminant. Then, the harvested differential equation does not have a real positive equilibrium. Therefore, there is no sustainable population size, and the crane population will decline to extinction.

**(b)** Solve the harvesting differential equation. When $t = 0$, we have $N = 194,600$. Then, find the minimum value of $t$ such that $N(t) < 0$. □

Example 3.8 (MA3264 AY25/26 Sem 1 Tutorial 6). It is found that near a certain seamount, but at a great depth, there is a population of at least 10 million *orange smoothies*, a kind of imaginary fish with a birth rate per capita estimated at $10^{-3}$ per year (they do not reproduce every year, so this is an average). After this discovery, fishing begins at a constant rate. What do you recommend? Use a logistic with harvesting model.

*Solution.* The differential equation is

$$\frac{dN}{dt} = 10^{-3}N - sN^2 - H.$$

At equilibrium, we set the derivative to be zero, so

$$N = \frac{10^{-3} \pm \sqrt{10^{-6} - 4sH}}{2s}.$$

Note that the carrying capacity is $K = 10,000,000$. Since $s = 10^{-3}/K$, then by the discriminant condition, we must have $10^{-6} - 4sH \geq 0$. Hence, $H \leq 2500$, so the maximum harvesting rate is 2500. This is very small compared to the general population of the fish which is around 10,000,000. $\square$

Example 3.9 (MA3264 AY25/26 Sem 1 Tutorial 6). Professor *Grospoisson*[2] studies a kind of fish which has a stable (against small perturbations) equilibrium population $P^+$. A major hurricane comes along and kills nearly all of the fish, but they are able to rebound after a while; but Prof G. finds to his surprise that the new population $P^-$, while stable, is smaller than $P^+$. He decides to model this strange situation with an equation of the form

$$\frac{dN}{dt} = -KN\left(N - P^+\right)\left(N - P^0\right)\left(N - P^-\right),$$

where $K$ and $P^0$ are positive constants, with $P^0$ equal to some number between $P^-$ and $P^+$, and $N(t)$ is the number of fish.

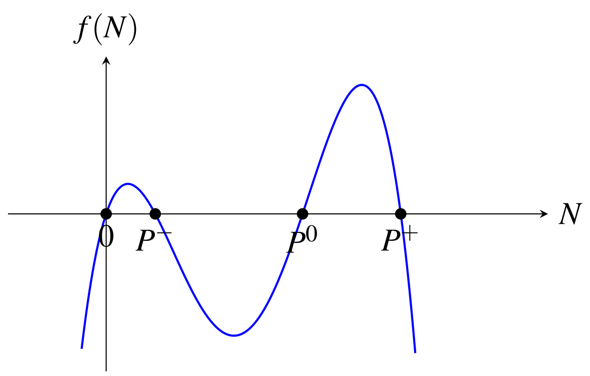



Figure 3.4: Graph of $y = -KN\left(N - P^+\right)\left(N - P^0\right)\left(N - P^-\right)$

However, Prof Grospoisson's young colleague, Dr. *Kleinerfisch*[3], claims that this model is wrong. He says that the minus sign in front of the right side should not be there. Who is correct?

[2]Most likely a play on the Mathematician Siméon Denis Poisson.
[3]Again, this is most likely a play on the Mathematicians Felix Klein and Ernst Sigismund Fischer.

*Solution.* We first note that $P^-$ and $P^+$ are distinct constants. Looking at the graph in Figure 3.4, one sees that the model predicts that there will be a stable equilibrium at two values of $N$. Note that for this to work, he does have to choose $P^0$ to be some number between $P^-$ and $P^+$. Since Dr. Kleinerfisch's model only has one stable equilibrium but the problem describes one with two stable equilibria, then Professor Grospoisson is correct. □

Example 3.10 (MA3264 AY25/26 Sem 1 Tutorial 6). In harvesting models, the population will rebound if all harvesting is stopped in time. Unhappily, this is not always true: for some animals, if you drive their population down too low, they will have trouble finding mates, or they will be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals, extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this *depensation*.

**(a)** Show that this situation can be modelled by the ordinary differential equation

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN \tag{3.2}$$

where $N$ is the population and $a, b, c$ are positive constants such that $b^2 > 4ac$.

**(b)** Find the population below which extinction will occur.

*Solution.*

**(a)** Note that the cubic polynomial $-aN^3 + bN^2 - cN$ passes through the origin. Since $a > 0$, then for large $N$, the graph tends to $-\infty$. In fact, the slope is negative on the right of the origin, so the derivative is $< 0$ for small values of $N$. As such, the differential equation (3.2) indeed models the mentioned situation.

**(b)** We have

$$\frac{dN}{dt} = -N\left(aN^2 - bN + c\right).$$

The roots of the cubic equation $-N\left(aN^2 + bN - c\right)$ are

$$N = 0 \quad N = \frac{b \pm \Delta}{2a} \text{ where } \Delta = \sqrt{b^2 - 4ac}.$$

Note that

$$N = \frac{b - \Delta}{2a}$$

is an unstable equilibrium. It is unstable because a population slightly above that will grow away from it, while a population slightly below it will drop towards zero. We see that the tigers will become extinct if the population ever falls below that value. □

Example 3.11 (MA3264 AY25/26 Sem 1 Tutorial 6). A constant harvesting rate $E$ can lead to disastrous results. Perhaps it would be better to use *flexible* harvesting, where for example the number of fish you catch depends on how many fish there are: you catch many when there are many fish, few when there are few fish. Suppose we replace $E =$ constant with a harvesting rate $E = \alpha N^2$, where $\alpha$ is a positive constant with the right units.

**(a)** Show that the fish population will eventually settle down to a stable equilibrium with this policy.

**(b)** Calculate the harvesting rate $R$ when the population is at equilibrium.

Notice that we have to choose $\alpha$ carefully — if you graph $R$ as a function of $\alpha$, you will find that it is small when $\alpha$ is small, but *also* when $\alpha$ is big.

**(c)** Find the optimal value of $\alpha$.

**(d)** What is the harvesting rate when that optimal value is chosen? Have you seen that number before?

*Solution.*

**(a)** The differential equation is

$$\frac{dN}{dt} = bN - sN^2 - \alpha N^2 = bN - (s+\alpha)N^2.$$

There is a stable equilibrium population at

$$\frac{B}{s+\alpha}.$$

Hence, every solution with $N(0) > 0$ tends to the positive equilibrium, so we conclude that the policy yields a stable equilibrium.

**(b)** The harvesting rate is $\alpha N^2$. At equilibrium, we have $N = \frac{B}{S+\alpha}$ so the harvesting rate is

$$\alpha\left(\frac{B}{S+\alpha}\right)^2 = \frac{\alpha B^2}{(S+\alpha)^2}.$$

**(c)** Let

$$f(\alpha) = \frac{\alpha B^2}{(S+\alpha)^2}.$$

Using differentiation, one can see that the unique maximum point is at $\alpha = s$.

**(d)** At the optimal $\alpha = s$, the harvesting rate is $B^2/4s$. This constant is precisely the critical value that appeared as the unsafe threshold under *constant* harvesting (recall Example 3.6). □

Chapter 4

# Systems of First-Order Differential Equations

## 4.1 Solving Systems of Ordinary Differential Equations

Relationships often go through ups and downs. We shall explore a mathematical model to capture this phenomenon. Romeo loves Juliet, but Juliet has a subtler response. When Romeo shows strong affection, Juliet finds his enthusiasm overwhelming, making her feelings for him cool down. However, when Romeo becomes indifferent, Juliet finds him mysteriously attractive. Romeo, on the other hand, reacts more directly: his love for Juliet increases when she is warm and decreases when she is cold.

Let $R(t)$ and $J(t)$ denote Romeo's and Juliet's feelings over time. These feelings can be modelled using the system of first-order linear ordinary differential equations as follows:

$$\frac{dR}{dt} = aJ \text{ and } \frac{dJ}{dt} = -bR \quad \text{where} \quad R(0) = \alpha \text{ and } J(0) = \beta.$$

Here, $a, b > 0$ are positive constants and $\alpha, \beta$ are initial feelings at $t = 0$. This system describes the interaction between their feelings.

We propose solutions of the form

$$R = Ae^{\lambda t} \quad \text{and} \quad J = Be^{\lambda t}.$$

Note that these can be obtained by transforming the system into two separate second-order linear differential equations, and then construct the characteristic equation to find the solution. Anyway, returning to the Romeo and Juliet problem, substituting $R$ and $J$ into the differential equations yields

$$A\lambda = aB \quad \text{and} \quad B\lambda = -bA.$$

Eliminating $A$ and $B$, we have $\lambda^2 = -ab$. Since $\lambda^2 < 0$, the solutions are complex, i.e. $\lambda = \pm i\sqrt{ab}$. As such, the general solution can be expressed as a linear combination of sin and cos as follows:

$$R = C\cos\left(\sqrt{ab}t\right) + D\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = E\cos\left(\sqrt{ab}t\right) + F\sin\left(\sqrt{ab}t\right)$$

All that is left is to find $C, D, E, F$. This can be done so by considering the initial conditions. As such,

$$R = \alpha\cos\left(\sqrt{ab}t\right) + \beta\sqrt{\frac{a}{b}}\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = \beta\cos\left(\sqrt{ab}t\right) - \alpha\sqrt{\frac{b}{a}}\sin\left(\sqrt{ab}t\right).$$

Motivated by the above, we consider a more general system, i.e.

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

We can write this as a matrix equation, which is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

We consider the solution

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{u}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

As such,

$$re^{rt}\mathbf{u}_0 = \mathbf{B}e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

or equivalently, $\mathbf{B}\mathbf{u}_0 = r\mathbf{u}_0$. This is analogous to the matrix equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, where $\lambda$ and $\mathbf{v}$ are an eigenvalue and corresponding eigenvector of the matrix $\mathbf{A}$! As such, the possibilities of $r$ are given by the eigenvalues of $\mathbf{B}$. We have

$$(\mathbf{B} - r\mathbf{I})\,\mathbf{u}_0 = \mathbf{0}$$

so non-trivial solutions exist if $\det(\mathbf{B} - r\mathbf{I}) = 0$, i.e. if $(a-r)(d-r) - bc = 0$. Except the case where this quadratic polynomial in $r$ has two repeated roots (i.e. discriminant zero), we must have two solutions $r_1$ and $r_2$, which implies

$$\mathbf{u} = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2,$$

where $c_1$ and $c_2$ are constants and $\mathbf{u}_1$ and $\mathbf{u}_2$ are the eigenvectors of $r_1$ and $r_2$ respectively. Naturally, $r_1$ and $r_2$ might be complex so we might have to interpret the exponential functions in terms of sine and cosine.

Example 4.1. Solve

$$\frac{dx}{dt} = -4x + 3y$$
$$\frac{dy}{dt} = -2x + y$$

*Solution.* In fact, such questions are also covered in MA3220. The matrix representation is

$$\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \quad \text{which} \quad \text{has eigenvalues } -1 \text{ and } -2.$$

The eigenspaces are

$$E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad E_{-2} = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}.$$

The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In other words,

$$x = c_1 e^{-t} + 3c_2 e^{-2t} \quad \text{and} \quad y = c_2 e^{-t} + 2c_2 e^{-2t}.$$

One can deduce the values of $c_1$ and $c_2$ given the values of $x(0)$ and $y(0)$. □

Example 4.2 (MA3264 AY25/26 Sem 1 Tutorial 7). A long and bitter battle is being fought on the slopes of Mount Doom between 15,000 Men of Gondor and 11,000 Orcs of Mordor (Figure 4.1). The Men die at a rate proportional to the number of Orcs, and also from a dread disease spread among them by the servants of Sauron, while the Orcs only die at a rate proportional to the number of Men — Orcs never get sick.

Let $G(t)$ denote the number of Gondorians and $M(t)$ denote the number of Mordor citizens in the battle. Then the above information tells us that we have a pair of differential equations which might have this form:

$$\frac{d}{dt}\begin{bmatrix} G \\ M \end{bmatrix} = \begin{bmatrix} -1 & -0.75 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} G \\ M \end{bmatrix},$$

where the $-1$ in the top left corner is the death rate per capita of the Gondorians due to disease, the $-0.75$ describes the rate at which Mordorians kill Gondorians, and the $-1$ in the second row describes the rate at which Gondorians kill Mordorians. Suppose the Men of Gondor are reinforced by 1000 warriors per day arriving from Rohan. The Orcs of Mordor are reinforced by 500 more Orcs arriving each day from the Dark Tower. Who wins now?


Figure 4.1: This problem was most likely inspired by Lord of the Rings

*Solution.* We have $G(0) = 15000$ and $M(0) = 11000$. Suppose

$$\frac{dG}{dt} = -G - 0.75M + 1000 \quad \text{and} \quad \frac{dM}{dt} = -G + 500.$$

Hence,

$$G = 500 - \frac{dM}{dt} \quad \text{so} \quad \frac{dG}{dt} = -\frac{d^2M}{dt^2}.$$

We obtain the differential equation

$$M'' + M' - \frac{3}{4}M = -500.$$

One can solve for $M$, then $G$, so we obtain

$$G(t) = 500 + 14750e^{-3t/2} - 250e^{t/2}$$

Set $G(t) = 0$. For this value of $t$, we see that $M(t)$ is still $> 0$, so the Orcs of Mordor win. $\square$

## 4.2 Classification using a Phase Plane

Consider a linear system of the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u} \quad \text{where} \quad \mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We classify the phase portrait by analysing the eigenvalues of the matrix $\mathbf{B}$. These are given by

$$r = \frac{1}{2}\left(\operatorname{tr}(\mathbf{B}) \pm \sqrt{(\operatorname{tr}(\mathbf{B}))^2 - 4\det(\mathbf{B})}\right).$$

Here, $\operatorname{tr}(\mathbf{B}) = a + d$ is the trace and $\det(\mathbf{B}) = ad - bc$ is the determinant. We now classify linear systems via their eigenvalues.

**(i) Real and distinct eigenvalues**

- Both positive implies we have a nodal source, i.e. trajectories move inwards, not spiralling
- Both negative implies we have a nodal sink, i.e. trajectories move outwards, not spiralling
- One positive and one negative eigenvalue implies we have a saddle point, i.e. trajectories approach along one eigenvector and escape along another

**(ii) Complex eigenvalues.** We write $r = \alpha \pm i\beta$ with $\alpha = \frac{1}{2}\operatorname{tr}(\mathbf{B})$.

- If $\alpha < 0$, we have a spiral sink so trajectories spiral inwards
- If $\alpha > 0$, we have a spiral source so trajectories spiral outwards
- If $\alpha = 0$, we have a centre, i.e. closed orbits with no decay or growth

We recall the Romeo and Juliet model in Chapter 4.1. Note that

$$\frac{d}{dt}\begin{bmatrix} R \\ J \end{bmatrix} = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}\begin{bmatrix} R \\ J \end{bmatrix}$$

where $a,b > 0$. Here, $\operatorname{tr}(\mathbf{B}) = 0$, $\det(\mathbf{B}) = ab > 0$, and the *discriminant* is $-4ab < 0$. Thus, the eigenvalues are purely imaginary. That is, $r = \pm i\sqrt{ab}$. Hence, the phase portrait is a centre — we have closed orbits in a clockwise direction, because along the positive $R$-axis, we have

$$\frac{dJ}{dt} = aR > 0.$$

Example 4.3 (MA3264 AY25/26 Sem 1 Tutorial 7). Classify the systems of ODEs with the following matrices. That is, say whether they represent a nodal source, spiral sink, etc.

**(a)**
$$\begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

**(b)**
$$\begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$$

**(c)**
$$\begin{bmatrix} -2 & -4 \\ 10 & 0 \end{bmatrix}$$

**(d)**
$$\begin{bmatrix} -5 & 4 \\ -2 & 1 \end{bmatrix}$$

**(e)**
$$\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$$

**(f)**
$$\begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix}$$

*Solution.*

**(a)** The eigenvalues are

$$\lambda = \frac{-1 \pm \sqrt{21}}{2}.$$

Since one eigenvalue is positive and the other is negative, we have a saddle point so trajectories approach along one eigenvector and escape along another.

**(b)** The eigenvalues are $\lambda = 1 \pm i\sqrt{7}$, so they are complex. Since the real part is $> 0$, we have a spiral source so trajectories spiral outwards.

**(c)** The eigenvalues are $\lambda = -1 \pm i\sqrt{39}$, so they are complex. Since the real part is $< 0$, we have a spiral sink so trajectories spiral inwards.

**(d)** The eigenvalues are $\lambda = -1, -3$ which are both negative. As such, we have a nodal sink so the trajectories move outwards, not spiralling.

**(e)** The eigenvalues are $\lambda = 1, 3$ which are both positive. As such, we have a nodal source so the trajectories move inwards, not spiralling.

**(f)** The eigenvalues are $\lambda = \pm i\sqrt{10}$, which are complex with real part 0. So, the system represents a centre, i.e. closed orbits with no decay or growth. □

Example 4.4 (MA3264 AY25/26 Sem 1 Tutorial 8). Classify the equilibria of the system

$$\frac{dx}{dt} = y - x \quad \text{and} \quad \frac{dy}{dt} = x(4-y).$$

*Solution.* Set $y - x = 0$ so $y = x$. Also, set $x(4-y) = 0$ so $x = 0$ or $y = 4$. Hence, the two equilibria are $(0,0)$ and $(4,4)$. The Jacobian matrix is

$$\mathbf{J}(x,y) = \begin{bmatrix} -1 & 1 \\ 4-y & -x \end{bmatrix}.$$

Then, compute the eigenvalues of the Jacobian matrix at $(0,0)$ and $(4,4)$. Use the above-mentioned classification to classify the two equilibria. □

Example 4.5 (MA3264 AY25/26 Sem 1 Tutorial 9). Romeo and Juliet are in a relationship governed by the equations

$$\frac{dR}{dt} = J \quad \text{and} \quad \frac{dJ}{dt} = -R + \varepsilon J^3,$$

where $\varepsilon$ is a small positive constant. Find the equilibrium solution here and classify it. Can you predict what will happen to the star-crossed lovers in the long run?

*Solution.* Set $\dot{R} = 0$ and $\dot{J} = 0$, which yield $J = 0$ and $\varepsilon J^3 = R$ respectively. The equilibrium solution is $(R,J) = (0,0)$. The Jacobian is

$$\mathbf{J}(R,J) = \begin{bmatrix} 0 & 1 \\ -1 & 3\varepsilon J^2 \end{bmatrix} \quad \text{so} \quad \mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues are $\pm i$, so the equilibrium point is a centre.

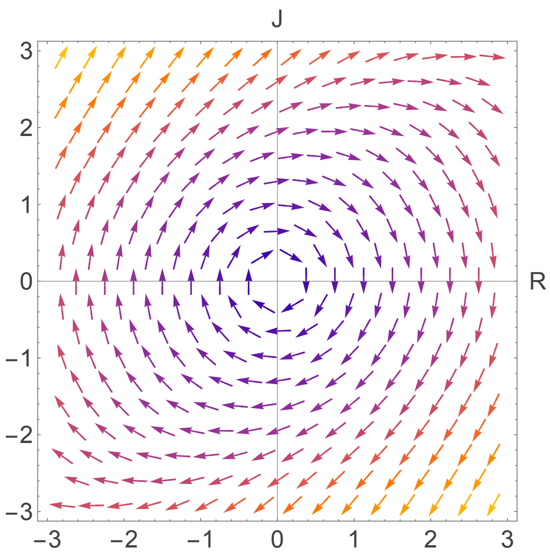



Figure 4.2: $\varepsilon = 0.1$

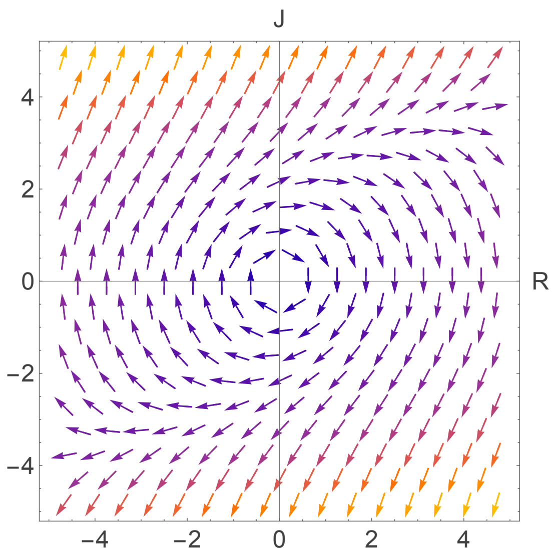



Figure 4.3: $\varepsilon = 0.1$ zoomed out

As for what happens in the long run, we cannot predict what will happen to the star-crossed lovers. At first glance, if we look at the phase plot of the differential equation over the region $x \in [-1,1], y \in [-1,1]$ (Figure 4.2), the system appears to form a center, which might suggest that Romeo and Juliet's relationship is stable. However, when we zoom out to the larger region $x \in [-3,3], y \in [-3,3]$ (Figure 4.3), the picture changes dramatically. The apparent stability was misleading: the dynamics reveal that it is extremely difficult to predict their long-term behavior. What becomes clear is that the relationship is not truly stable. Instead, it spirals outward and ultimately explodes in a tragic and unpredictable fashion. □

Example 4.6 (MA3264 AY25/26 Sem 1 Tutorial 7). King Xerxes I of Persia has sent a million soldiers to conquer Greece. King Leonidas I of Sparta decides to meet the Persians at Thermopylae. A typical Persian soldier can kill one Spartan per hour, whereas a typical Spartan soldier can kill $11,111,111.1$ Persians per hour. Neither side suffers from any disease.[1] How many soldiers does King Leonidas need to take to Thermopylae if he wants to kill all of the Persians but expects no Spartans to go home alive?

*Solution.* Let $P(t)$ denote the number of Persians at time $t$ and $S(t)$ denote the number of Spartans at time $t$. Then,

$$\frac{dS}{dt} = -P$$

since each Persian kills one Spartan per hour. On the other hand,

$$\frac{dP}{dt} = -11,111,111.1S.$$

Hence,

$$\frac{d^2S}{dt^2} = 11,111,111.1S.$$

[1] They all die before they have time to get sick.

One sees that the solution to this second-order differential equation is

$$S=c_1e^{\frac{10^4}{3}t}+c_2e^{-\frac{10^4}{3}t}.$$

So,

$$P=-\frac{10^4}{3}c_1e^{\frac{10^4}{3}t}+\frac{10^4}{3}c_2e^{-\frac{10^4}{3}t}.$$

As $t$ tends to infinity, we must have $P$ tends to 0, so $c_1=0$. Since $P(0)=10^6$, then $c_2=300$. Hence, $S(0)=300$ as well. □

Example 4.7 (MA3264 AY25/26 Sem 1 Tutorial 8). Suppose

$$\frac{d\mathbf{r}}{dt}=\mathbf{B}\mathbf{r}$$

is a system of linear first-order differential equations with coefficient matrix **B**.

**(i)** Show that $\det(\mathbf{B})<0$ if and only if the system is a saddle. This is rather useful and worth remembering.

**(ii)** Show that if you know that

$$(\operatorname{tr}(\mathbf{B}))^2-4\det(\mathbf{B})>0 \quad \text{and} \quad \det(\mathbf{B})>0$$

then you have a node.

*Solution.*

**(i)** Recall when we were classifying the linear system that if the matrix has real and distinct eigenvalues such that one eigenvalue is positive and the other is negative, then we obtain a saddle point. Since the determinant of a matrix **B** is equal to the product of its eigenvalues, the result follows.

**(ii)** Let

$$\mathbf{B}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then,

$$(\operatorname{tr}(\mathbf{B}))^2-4\det(\mathbf{B})=(a+d)^2-4(ad-bc)=(a-d)^2+4bc>0$$

and

$$\det(\mathbf{B})=ad-bc>0.$$

To prove that we have a node, it suffices to show that both eigenvalues are real and positive, or real and negative. Since $(a-d)^2+4bc>0$ denotes some discriminant, then the roots are real. The determinant property tells us that the roots are either both positive or both negative. □

Example 4.8 (MA3264 AY25/26 Sem 1 Tutorial 9). Let $\varepsilon > 0$ be a positive constant. Show that, in the Van der Pol oscillator equations

$$\frac{dx}{dt} = y \quad \text{and} \quad \frac{dy}{dt} = -x + \varepsilon\left(1 - x^2\right) y,$$

the equilibrium at the origin is only a spiral source if $\varepsilon < k$, for some constant $k$. Find $k$. What happens if $\varepsilon > k$? Explain why the local analysis is misleading.

*Solution.* Set $\dot{x} = 0$ and $\dot{y} = 0$. Then, $y = 0$ and $-x + \varepsilon\left(1 - x^2\right) y = 0$. So, $x = 0$. The Jacobian matrix is

$$\mathbf{J}(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon\left(1 - x^2\right) \end{bmatrix} \quad \text{so} \quad \mathbf{J}(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}.$$

The eigenvalues of the matrix are

$$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}.$$

For the equilibrium to be a spiral source, the eigenvalues must be complex with a positive real part. So, we must have $\varepsilon^2 - 4 < 0$. Hence, $k = 2$. If $\varepsilon > k = 2$, then we obtain two real and distinct eigenvalues. Note that both eigenvalues are positive, so we obtain a nodal source. The classification is technically correct locally, but the global picture is very different: regardless of whether it is a spiral or node, the origin is unstable and the long-term behaviour is governed by a limit cycle, not by the linearisation. □

Example 4.9 (MA3264 AY25/26 Sem 1 Tutorial 7). Both Elves and Dwarves live in Rivendell, but there is a certain resentment between the two groups. The amount of racial prejudice is not the same on both sides, Elves tending to be more intolerant than Dwarves, but both groups are inclined to move out of Rivendell at a rate controlled by the number of the other group. Apart from this, both groups reproduce as usual, but Elves can only be killed by violence — they never grow old, so their overall death rate per capita is lower than that of the Dwarves; also, the Elvish birth rate per capita is higher since Dwarf women are scarce. Both groups have more births than deaths. We can set up a model of this situation using a pair of simultaneous ordinary differential equations with a matrix

$$\begin{bmatrix} 5 & -4 \\ -1 & 2 \end{bmatrix}$$

where the first row describes the rate of change of the Elf population.

**(a)** Explain why this matrix does represent the situation we described.

**(b)** At a certain time, the number of Dwarves is slightly larger than the number of Elves. Predict what will happen to the population of Elves. This pattern has in fact often been observed in human societies.

*Solution.*

**(a)** We have

$$\dot{E} = 5E - 4D \quad \text{and} \quad \dot{D} = -E + 2D.$$

The coefficient 5 in front of $E$ means Elves have a strong intrinsic growth, and the coefficient 2 in front of $D$ means Dwarves also grow but more slowly. The $-4D$ in the Elf equation means the more Dwarves there are, the more pressure Elves feel to leave, reflecting prejudice and resentment. The $-E$ in the Dwarf equation means that Elves also push Dwarves out but less severely. Indeed, this matches the story.

**(b)** Suppose $D$ is only slightly larger than $E$. Then, there exists some small $\delta > 0$ such that $D = E + \delta$. Then,

$$\dot{E} = 5E - 4(E + \delta) = E - 4\delta.$$

Unless $E$ is very large compared to $\delta$, this quantity is negative. So, whenever Dwarves slightly outnumber Elves, the Elf population starts to decrease. The Elf population is driven downward once they are outnumbered. This reflects real social dynamics: a minority group facing stronger intolerance often declines or emigrates, even if they have advantages in reproduction and longevity. □

Introducing constant reinforcements $(g, m)$, we obtain the differential equation

$$\frac{d\mathbf{r}}{dt} - \mathbf{B}\mathbf{r} = \mathbf{F}$$

whose equilibrium point is shifted to

$$\mathbf{r}_{\text{equilibrium}} = -\mathbf{B}^{-1}\mathbf{F}.$$

The phase portrait is translated by this vector, but its qualitative shape remains exactly the same. Thus, by choosing reinforcement values cleverly, one may shift the saddle point so that the initial state lies on the favourable side of the separatrix.

Chapter 5

# Modelling with Non-Linear Systems

## 5.1 The Lotka-Volterra Model

Lions like to eat zebras, and depend on them. That is, the lion population goes up if there are many zebras. However, if there were no zebras, then the lions would die out. The zebras eat grass, and would get along just fine if there were no lions. Their population tends to go down when there are lions about but when left to themselves, their population goes up.

Suppose at time $t$, there are $L(t)$ lions and $Z(t)$ zebras, and assume a Malthusian model for both the lions and zebras in the absence of the other. We shall assume that there is a stab equilibrium at populations $(L_0, Z_0)$. This suggests that we can devise the following model:

$$\begin{aligned}
\frac{dL}{dt} &= -(L-L_0) + 2(Z-Z_0) \\
\frac{dZ}{dt} &= -2(L-L_0) + \frac{1}{2}(Z-Z_0)
\end{aligned}$$

This resembles the models mentioned in the previous chapter, except that the equilibrium has been shifted from $(0,0)$ to a point in the first quadrant. One can verify that in this case, the equilibrium point is indeed $(L_0, Z_0)$. In fact, it is a type of stable equilibrium — it is a spiral sink. If there is some kind of disturbance, the populations of lions and zebras fluctuate up and down for a while but they eventually get close to equilibrium.

We shall further analyse this model. Suppose $L(0) = 0$ and $Z(0) = 0$. Then, when $t = 0$, $dL/dt = L_0 - 2Z_0$ which is non-zero. The same can be said for $dZ/dt$ when evaluated at $t = 0$. This means that lions and zebras are coming into existence out of nothing or that we will immediately have negative numbers of animals! Moreover, the system we are trying to model has two equilibria — other than $(L_0, Z_0)$, we also have $(0,0)$. That is, it must always be possible to have no lions and no zebras. However, this is not what we will get or with any linear model as such systems only ever have one equilibrium point.

We shall construct a different mathematical model for the lion and zebra situation. This is similar to the logistic model we discussed previously in a sense that the death rate per capita of zebras, $D_Z$, is not fixed. Here, $D_Z$ depends on the number of lions, so suppose

$$D_Z = sL \quad \text{where} \quad s \text{ is a positive constant.}$$

The constant $s$ tells us something about the relationship between lions and zebras. We continue to assume a Malthusian model for the zebra birth rate per capita, $B_Z$. As such, we have

$$\frac{dZ}{dt} = B_Z Z - sLZ.$$

What about the lions? When there is a shortage of zebras, they can eat other animals so they will not really starve. However, zebras are *nice and fat*, so the ones which really suffer from a shortage of zebras is not the adult lions but rather, the baby lions. This is because if there is an insufficient number of *nice and fat* zebras around, then the mother lion cannot produce enough milk for the young, and the latter will die. As such, the effect of a shortage of zebras is to reduce the effective birth rate of the lions. Hence, we use a Malthusian model for the death rate of the lions. We write

$$\frac{dL}{dt} = uZL - D_L L.$$

The pair of equations

$$\frac{dZ}{dt} = B_L Z - sLZ \quad \text{and} \quad \frac{dL}{dt} = uZL - D_L L$$

gives a famous model of such populations known as the Lotka-Volterra model, or the predator-prey model. One verifies that $(L,Z) = (B_Z/s, D_L/u)$ is an equilibrium point, and so is $(L,Z) = (0,0)$! As such, we are on the right track. However, the Lotka-Volterra equations are non-linear!

Example 5.1 (MA3264 AY25/26 Sem 1 Tutorial 7). An apple farm is affected by bugs (called mites) that attack trees. They are kept in control by a different kind of predatory insect which eat them. Let $M(t)$ be the number of mites and $P(t)$ be the number of predators.

**(i)** Set up a Lotka-Volterra model to describe this situation.

Now, the farmer, fed up with the bugs, sprays his apple trees with a chemical which kills the mites at a per capita rate of $x$ per week. Of course, the predators also die: let us assume that the chemical also kills them at the same per capita rate $x$ per week.

**(ii)** What happens to the equilibrium populations? This problem is known as the paradox of the pesticides, which states that applying pesticide to a pest may end up increasing the abundance of the pest or other pests if the pesticide upsets natural predator-prey dynamics in the ecosystem.

*Solution.*

**(i)** We have
$$\frac{dM}{dt} = rM - aMP \quad \text{and} \quad \frac{dP}{dt} = bMP - dP.$$
Here, we let $r > 0$ be the intrinsic growth rate of mites, $a > 0$ denote its predation rate, $d > 0$ denote the death rate of predators, and $b > 0$ denote the conversion efficiency of consumed mites into predators.

**(ii)** We first find the original equilibrium positions. Setting $M' = 0$ yields $r = aP$, and setting $P' = 0$ yields $bM = d$. So,
$$(M^*, P^*) = \left(\frac{d}{b}, \frac{r}{a}\right).$$
For the new setup, each population acquires an additional mortality $x$. Equivalently, $r \mapsto r - x$ and $d \mapsto d + x$ which yields
$$M' = (r-x)M - aMP \quad \text{and} \quad P' = bMP - (d+x)P$$
The new equilibrium positions are
$$M^* = \frac{d+x}{b} \quad \text{and} \quad P^* = \frac{r-x}{a}.$$
This is biologically feasible if $r - x > 0$, or equivalently $0 < x < r$. So, as the pesticide level $x$ increases (but kept in the interval $0 < x < r$), the equilibrium mite level increases while the predator level decreases. This is precisely the paradox of the pesticides — killing both species at the same per-capita rate pushes the predator nullcline rightward and the prey nullcline downward. □

Example 5.2 (MA3264 AY25/26 Sem 1 Tutorial 8). Hares in Canada have a birth rate per capita of 1 per month. They are devoured by lynx, a kind of large cat, at a rate per capita equal to the population of lynx. The lynx have an effective birth rate per capita equal to the number of hares, and a death rate per capita of 1 per month. At a certain point in time, there are 2000 hares and 2000 lynx in a particular forest. Write down an exact equation relating the number of hares (in thousands) at any time, $H(t)$, to the number of lynx (in thousands) at that time, $L(t)$, assuming that neither population is ever equal to zero.

*Solution.* The Lotka-Volterra equations are
$$\frac{dH}{dt} = H - HL \quad \text{and} \quad \frac{dL}{dt} = HL - L.$$
So,
$$\frac{dH}{dL} = \frac{H - HL}{HL - L} = \frac{H(1-L)}{L(H-1)} \quad \text{which implies} \quad \frac{H-1}{H}\,dH = \frac{1-L}{L}\,dL.$$
Integrating both sides yields
$$H - \ln|H| = \ln|L| - L + c.$$
Since $(H, L) = (2, 2)$ denotes the initial condition, substituting into the solution yields $c = 4 - 2\ln 2$ so we conclude that $H + L - \ln H - \ln L = c - 2\ln 2$. □

Example 5.3 (MA3264 AY25/26 Sem 1 Tutorial 8). Lions like to eat zebras. The lions have a constant death rate per capita of 20% per year, and their effective birth rate per capita is $B_L = uZ$, where $Z$ is the number of zebras and $u$ is a constant equal to about 0.0008 per year. Meanwhile, the zebras have a constant birth rate per capita of about 5% per year, and their death rate per capita is $D_Z = sL$, where $s$ is a constant approximately equal to 0.004 per year, and $L$ is the number of lions. Compute the equilibrium population of lions and zebras.

*Solution.* We have

$$\frac{dL}{dt} = 0.0008LZ - 0.2L \quad \text{and} \quad \frac{dZ}{dt} = 0.05Z - 0.004ZL.$$

Setting $L' = 0$ and $Z' = 0$ yield the equilibrium populations. We have $Z^* = 250$ and $L^* = 12.5$. □

## 5.2 Linearisation

Recall the Taylor series expansion for functions of several variables from MA2104/MA3210. Now, suppose that we have a pair of non-linear simultaneous first-order ODEs governing a pair of functions of time $(x(t), y(t))$ of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

and suppose that this system is known to have an equilibrium point at $(x,y) = (a,b)$. This implies $f(a,b) = g(a,b) = 0$. As such, when we obtain the Taylor series expansion for these two functions around the point $(a,b)$, the constant term vanishes! Keeping the linear terms and discarding terms of higher order, we obtain the following equations:

$$\begin{aligned} \frac{dx}{dt} &\approx f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ \frac{dy}{dt} &= g_x(a,b)(x-a) + g_y(a,b)(y-b) \end{aligned}$$

which we can write as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

and now, the equations have become linear! One recalls from MA2104/ST2131 that the matrix of partial derivatives here is known as the Jacobian matrix. In summary, near an equilibrium point, a non-linear system can be approximated by a certain linear system with a matrix given by the Jacobian of the original system. This new system is called the linearisation of the original system. Indeed, this is a good piece of news as we can now apply knowledge from the previous chapter to solve our new pair of differential equations! In particular, the classification theorem of equilibrium points (recall MA3220) now applies. As such, we still have a good idea of what is happening in the phase diagram near to those points — we just have to compute the Jacobian there.

Example 5.4 (lions and zebras). Now, recall the Lotka-Volterra equations for the lion and zebra problem. That is,

$$\frac{dL}{dt} = uZL - D_L L = f(L,Z)$$
$$\frac{dZ}{dt} = B_Z Z - sLZ = g(L,Z)$$

The Jacobian matrix here is

$$\mathbf{J}(L,Z) = \begin{bmatrix} uZ - D_L & uL \\ -sZ & B_Z - sL \end{bmatrix}$$

and we wish to evaluate it at the two equilibrium points.

The first is $(L,Z) = (0,0)$ and so

$$\mathbf{J}(0,0) = \begin{bmatrix} -D_L & 0 \\ 0 & B_Z \end{bmatrix}.$$

It is easy to see that this is a saddle point. Since the matrix $\mathbf{J}(0,0)$ is diagonal, the eigenvalues are the diagonal entries, namely $-D_L$ and $B_Z$, with corresponding eigenvectors $(1,0)$ and $(0,1)$ respectively. One checks that this makes sense because in the phase plane, everything is rushing towards the origin along the lion axis and away from the origin along the zebra axis. We can expect that if we start nearer to the $L$-axis, then the lion population will decrease greatly until the zebra populaltion increases rapdily, which makes sense!

The other equilibrium point is $(L,Z) = (B_Z/s, D_L/u)$, and one checks that

$$\mathbf{J}\left(\frac{B_Z}{s}, \frac{D_L}{u}\right) = \begin{bmatrix} 0 & uB_Z/s \\ -sD_L/u & 0 \end{bmatrix}$$

which we recognise as a centre. As such, in the middle of the phase diagram corresponding to large numbers of both lions and zebras, we expect to see the swirling motion. In fact, the direction of motion is clockwise.

Now, take the two Lotka-Volterra equations, multiply the $dL/dt$ equation by $B_Z/L - s$, and multiply the $dZ/dt$ equation by $D_L/Z - u$. Adding the resulting equations, we have

$$\left(\frac{B_Z}{L} - s\right)\frac{dL}{dt} + \left(\frac{D_L}{Z} - u\right)\frac{dZ}{dt} = 0.$$

Hence,

$$B_Z \ln L - sL + D_L \ln Z - uZ = c,$$

where $c$ is an arbitrary constant. This is an exact relation between $Z$ and $L$ although the equation cannot be explicitly solved. One way to draw the graphs is to define a function $F(L,Z)$ on the phase plane by

$$F(L,Z) = B_Z \ln L - sL + D_L \ln Z - uZ.$$

One checks thatt this function has a global minimum at the point $(L,Z) = (B_Z/s, D_L/u)$, with contour curves around that point which are all closed. Since the paths in the phase plane are all closed curves, we see that all solutions of the Lotka-Volterra equations are periodic.

The Lotka-Volterra model can be used to understand an interesting paradox known as the paradox of pesticides. This is the strange observation that when a certain pest has a predator, using pesticides can actually leads to more pests than we had initially!

## 5.3 Logistic Lotka-Volterra Model

The Lotka-Volterra model assumes that the zebra population grows according to a Malthusian model when there are no lions. We know that this is not realistic so we should use something like the logistic model for them, while keeping the old equation for the lions. As such, we obtain the following pair of differential equations:

$$\frac{dL}{dt} = uZL - D_L L$$
$$\frac{dZ}{dt} = B_Z Z - pZ^2 - sLZ$$

Here, $p$ is the logistic constant, so the equilibrium population of zebras would be $B_Z/p$ in the complete absence of lions.

In this model, there are actually three equilibrium points in the phase diagram. The first is the obvious one $(0,0)$. The second is almost as obvious, i.e. if there are no lions, then the zebras will approach a logistic equilibrium along the $Z$-axis, i.e. the point $(0, B_Z/p)$. The third and most interesting one is at

$$\left( \frac{B_Z - pD_L/u}{s}, \frac{D_L}{u} \right).$$

We omit the remaining details.

Example 5.5 (MA3264 AY25/26 Sem 1 Tutorial 8). Consider the lions and zebras in Example 5.3, and assume that we allow hunting at an outrageous rate of 10 lions per year. Classify the equilibrium point in this case and predict the outcome. To verify your assertion, you may consider say

$$\frac{dL}{dt} = 0.0008LZ - 0.2L - 10 \quad \text{and} \quad \frac{dZ}{dt} = 0.05Z - 0.004LZ.$$

*Solution.* Set $L' = 0$ and $Z' = 0$. In Example 5.3, we saw that the original equilibrium populations are $Z^* = 250$ and $L^* = 12.5$. We see that the new non-trivial equilibrium populations are $Z^* = 1250$ and $L^* = 12.5$ (the other equilibrium population for $L^* < 0$ which is not meaningful biologically). From our knowledge of linearisation in Chapter 5.2, let

$$f(L,Z) = 0.0008LZ - 0.2L - 10 \quad \text{and} \quad g(L,Z) = 0.05Z - 0.004LZ.$$

So,

$$\mathbf{J}(L,Z) = \begin{bmatrix} 0.0008Z - 0.2 & 0.0008Z \\ -0.004Z & 0.05 - 0.004L \end{bmatrix}.$$

Hence,

$$\mathbf{J}(12.5, 1250) = \begin{bmatrix} 0.8 & 0.01 \\ -5 & 0 \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix are real and positive, which suggest a nodal source.

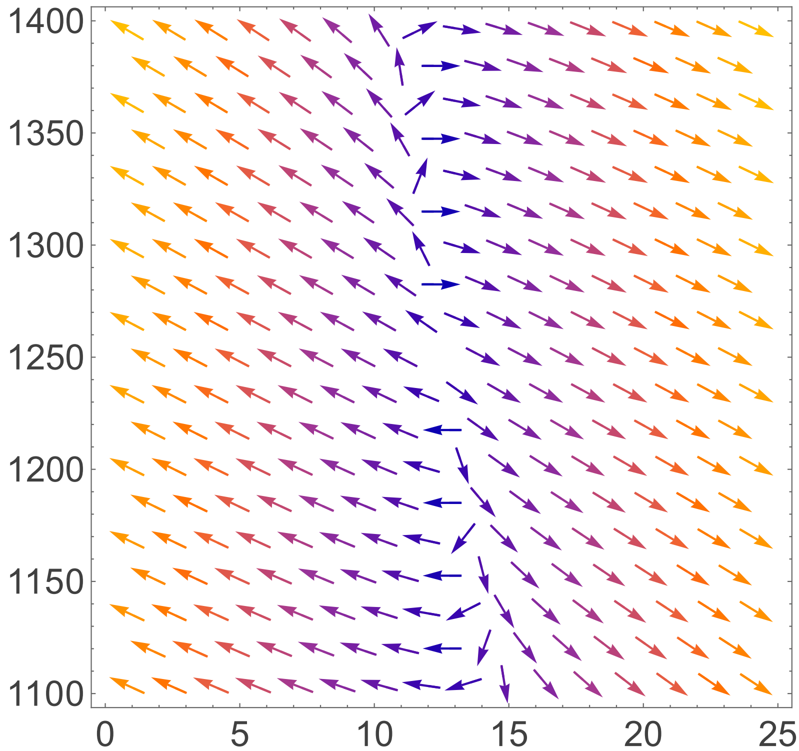



Figure 5.1:

That is to say, trajectories move inward and do not spiral as shown in Figure 5.1. □

Example 5.6 (MA3264 AY25/26 Sem 1 Tutorial 9). The Etosha National Park in Namibia has lions and zebras. One day the lions all die because of excessive hunting, and the zebras settle down to a logistic equilibrium population of 10000. The Namibian authorities decide to ban all hunting and try again: they import some lions from neighbouring Botswana. The lion death rate per capita is 10% per year, and each zebra raises the effective birth rate per capita of the lions by 0.002%.

**(i)** If each female zebra gives birth to 4 babies per year, and the equally numerous male zebras give birth to a negligible number, and if one lion increases the zebra death rate per capita by 1%, what will the eventual population of zebras be, in a logistic Lotka-Volterra model?

**(ii)** How many lions will there be in Etosha eventually?

*Solution.*

**(i)** Let $L(t)$ and $Z(t)$ denote the lion and zebra population respectively. We have

$$\frac{dL}{dt} = 0.00002LZ - 0.1L \quad \text{and} \quad \frac{dZ}{dt} = rz\left(1 - \frac{Z}{K}\right) - sLZ.$$

Here, $r$ is the zebra per capita birth rate, $K$ is the zebra carrying capacity in the absence of lions (here, we are assuming a Malthusian growth model), $s$ is the zebra death rate per lion. We see that $K = 10000$. We also have $r = 2$ (since half of the babies are female). Also, $s = 0.01$. The Lotka-Volterra equations are

$$\frac{dL}{dt} = 0.00002LZ - 0.1L \quad \text{and} \quad \frac{dZ}{dt} = 2Z\left(1 - \frac{Z}{10000}\right) - 0.01LZ.$$

Set $Z' = 0$ so obtain

$$2Z - \frac{Z^2}{5000} - 0.01LZ = 0.$$

Assuming the equilibrium population to be positive, we have the quadratic equation $10000 - Z - 50L = 0$. Setting $L' = 0$ yields either $L = 0$ or $0.00002Z = 0.1$ so $L^* = 5000$. That is, the eventual zebra population is 5000.

**(ii)** Previously, we had $10000 - Z - 50L = 0$, where $Z = 5000$. So, $L = 100$.

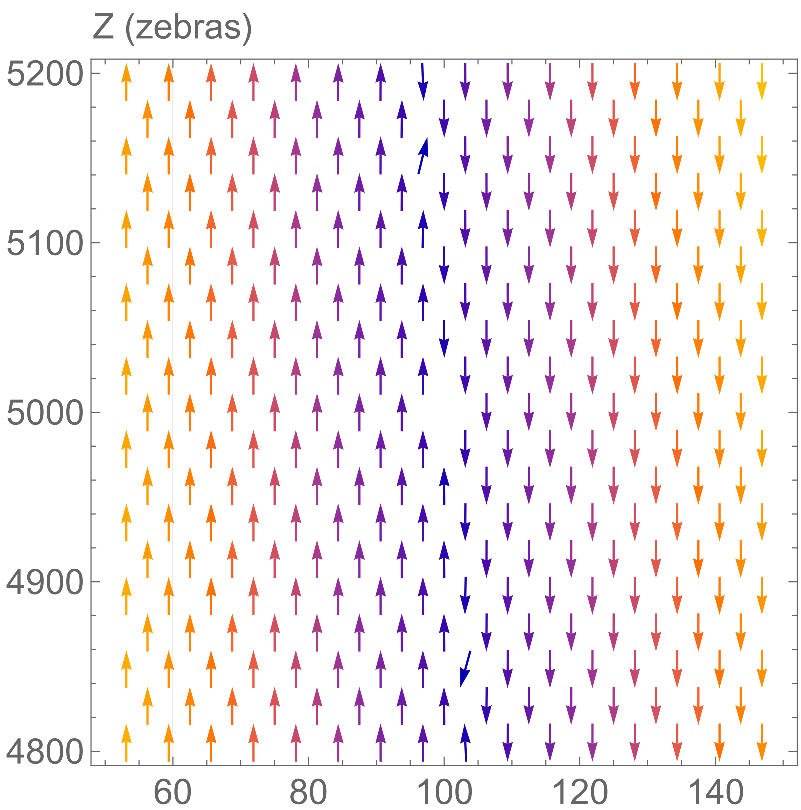



Figure 5.2:

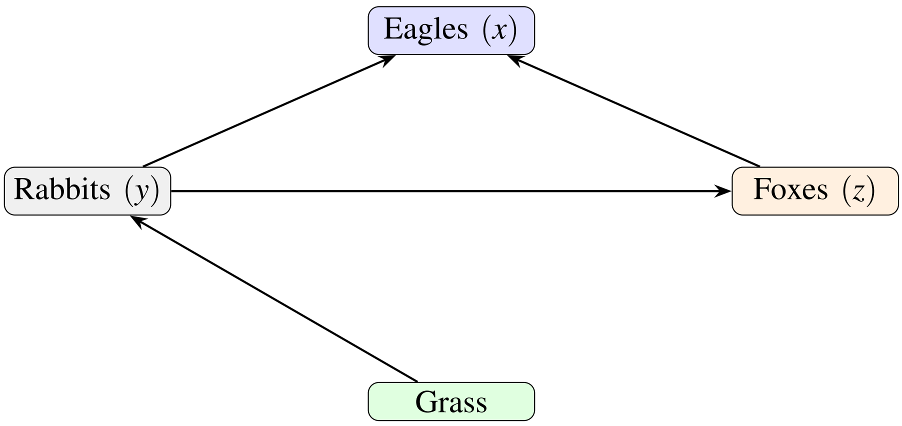
In fact, we see from Figure 5.2 that the equilibrium point is a spiral. □

Example 5.7 (MA3264 AY25/26 Sem 1 Tutorial 8). Foxes eat rabbits. Rabbits eat grass. Eagles eat both foxes and rabbits. Let $x(t)$ be the number of eagles at time $t$, let $y(t)$ be the number of rabbits, and let $z(t)$ be the number of foxes, all in thousands. Argue that the following equations could describe this situation:

$$\dot{x} = -x + xy + xz \quad \dot{y} = y - xy - yz \quad \dot{z} = -z + yz - xz$$

At time $t = 0$ it is observed that there are 4 thousand rabbits, 2 thousand foxes, and 1 thousand eagles in a certain national park. Predict the number of foxes in the long run.

*Solution.* Consider the following food chain, which makes sense because eagles die naturally and they increase in proportion to successful hunts of rabbits and foxes, rabbits grow logistically at baseline but are eaten by eagles and foxes, foxes die naturally and they grow when eating rabbits but they are eaten by eagles.




Set $\dot{x} = 0$. Then, $x = 0$ or $y + z = 1$. Set $\dot{y} = 0$. Then, $y = 0$ or $x + z = 1$. Set $\dot{z} = 0$. Then, $z = 0$ or $y - x = 1$. We shall look for positive $x, y, z$. If $x \neq 0$, $y \neq 0$, $z \neq 0$, we simultaneously need $y + z = 1$, $x + z = 1$, $y - x = 1$. However, we see that the system has no solution, so there is no strictly positive interior equilibrium.

We check the cases where one population vanishes. If $x = 0$, then $\dot{y} = y(1 - z)$ and $\dot{z} = z(y - 1)$. The only stable equilibrium is $(0, 1, 0)$, i.e. only rabbits survive. For the other long-term equilibria, we see that they are not meaningful because the rabbit population is zero. So, in the long run, the fox population goes to zero. Rabbits stabilize at 1 thousand, eagles die out (only rabbits remain). □

Example 5.8 (MA3264 AY25/26 Sem 1 Tutorial 9). Previously, we considered lion hunting in the case of the Lotka-Volterra model. Let us see what happens if we hunt lions (at a constant rate), beginning with the *logistic* Lotka–Volterra equations

$$\frac{dL}{dt} = uZL - D_L L - H \quad \text{and} \quad \frac{dZ}{dt} = B_Z Z - sLZ - pZ^2.$$

There are several different possibilities here. Consider the special case $u = s = p = 1$, $B_Z = 5$, $D_L = 2$, $H = 2$. Show that there are two equilibria, one unstable, one stable. So the lions do not necessarily become extinct here, though they easily could. On the other hand, it is easy to ensure that the lions will be wiped out, if that is really what you want: what happens if $p = 2$ instead of 1?

*Solution.* The Lotka-Volterra equations become

$$\frac{dL}{dt} = ZL - 2L - 2 \quad \text{and} \quad \frac{dZ}{dt} = 5Z - LZ - Z^2.$$

Set $L' = 0$ and $Z' = 0$ so

$$ZL - 2L - 2 = 0 \quad \text{and} \quad Z(5 - L - Z) = 0.$$

If $Z = 0$, then $L = -1$. However, this is nonsensical since the population cannot be negative. Consider $5 - L - Z = 0$. So, $Z = 5 - L$. We obtain $L(5-L) - 2L - 2 = 0$, so $L = 1$ or $L = 2$. If $L = 1$, then $Z = 4$; if $L = 2$, then $Z = 3$.

We have the Jacobian matrix

$$\mathbf{J}(L,Z) = \begin{bmatrix} Z-2 & L \\ -Z & 5-L-2Z \end{bmatrix}.$$

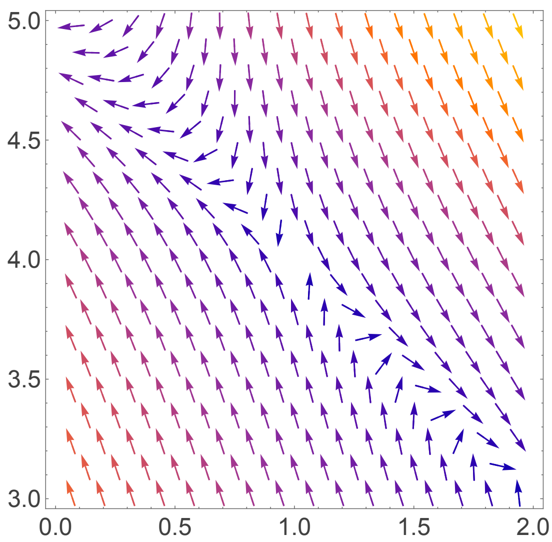


Figure 5.3:

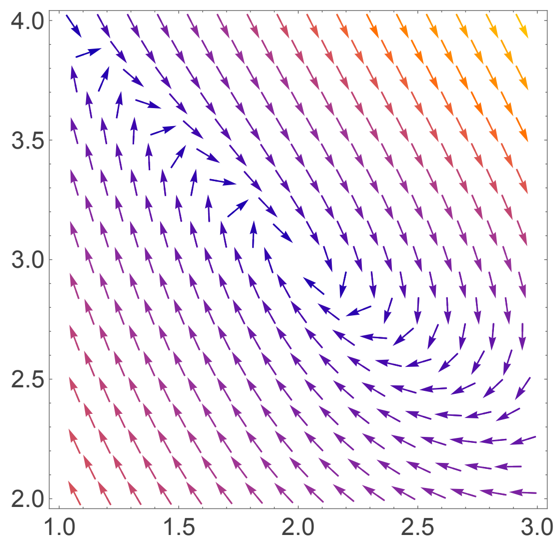


Figure 5.4:

So,

$$\mathbf{J}(1,4) = \begin{bmatrix} 2 & 1 \\ -4 & -4 \end{bmatrix}.$$

The eigenvalues are $\lambda = -1 \pm \sqrt{5}$, so one is positive and one is negative. So, $(1,4)$ is a saddle point (Figure 5.3).

Next,

$$\mathbf{J}(2,3) = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix}$$

which has eigenvalues $\lambda = -1 \pm \sqrt{2}i$. Since the real part is negative, we obtain a spiral sink. That is, $(2,3)$ is a spiral sink (Figure 5.4). $\square$

Example 5.9 (MA3264 AY25/26 Sem 1 Tutorial 10). Very often, in real populations in Africa, zebras enormously outnumber lions.

**(i)** The following equations might model such a situation. Why?

$$\frac{dL}{dt} = \left(\frac{Z}{Z+1}\right)L - 0.5L, \qquad \frac{dZ}{dt} = Z - 0.5Z^2 - \left(\frac{Z}{Z+1}\right)L. \tag{5.1}$$

**(ii)** Find the equilibrium in the case where there are non-zero numbers of both lions and zebras. Is it stable?

**(iii)** Now, suppose fertilizers increase grass growth, doubling the effective birth rate of zebras while all other parameters remain the same. There is now a new equilibrium. Is it stable? What happens?

*Solution.*

**(i)** Consider the Lotka-Volterra equations as in (5.1). For the zebra population, $Z - 0.5Z^2$ is the logistic growth with intrinsic growth rate $r = 1$ and carrying capacity $K = 2$. Consider the proportion $\frac{Z}{Z+1}$ — when zebras are abundant, each lion's intake saturates near 1; when zebras are scarce, the intake is approximately $Z$.

**(ii)** Let $L' = 0$ and $Z' = 0$. Then,

$$L\left(\frac{Z}{Z+1} - 0.5\right) = 0 \quad \text{and} \quad Z - 0.5Z^2 - \left(\frac{Z}{Z+1}\right)L = 0.$$

Since we are interested in a non-zero equilibrium population, one can easily deduce it is $(L^*, Z^*) = (1,1)$. The Jacobian matrix is

$$\mathbf{J}(L,Z) = \begin{bmatrix} \frac{Z}{Z+1} - \frac{1}{2} & \frac{L}{(Z+1)^2} \\ -\frac{Z}{Z+1} & 1 - Z - \frac{L}{(Z+1)^2} \end{bmatrix}.$$

At $(1,1)$,

$$\mathbf{J}(1,1) = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

which has complex eigenvalues with negative real part. So, $(1,1)$ is a spiral sink.

**(iii)** Now,

$$\frac{dZ}{dt} = 2Z - 0.5Z^2 - \frac{Z}{Z+1}L$$

whereas $\dot{L}$ is unchanged. We omit the solution as it is too trivial. □

## 5.4 Competing Species

Instead of studying animals eating one another, we can use these methods to study them competing. Take for example in Australia where there are no wolves but originally, there was an animal that played much the same role, the thylacine, also known as the *Tasmanian Tiger*. Around 4000 years ago, a similar animal, the dingo, was introduced to the mainland. Shortly after its arrival, the Tasmanian Tiger disappeared from the mainland (though it survived in Tasmania until 1936). One possible, though debated, explanation is that the thylacine was driven to extinction through competition with the dingo. We now construct a mathematical model to explore this hypothesis.

We assume a logistic model for the thylacines in the absence of the dingo. That is,

$$\frac{dT}{dt} = aT - bT^2.$$

Now, the dingo supposedly competed for food with the thylacine, so mother thylacine had less milk. This implies that the effective birth rate per capita was driven down by the dingo. We try $B_T = a - kD$, where $D$ is the number of dingoes. As usual, $k$ measures the effect on the thylacine birth rate per capita of the arrival of one dingo. We do not assume any effect on the death rate per capita. We will also assume that the thylacine had a similar effect on the dingo birth rate per capita. As such, for positive constants $c, \sigma, d$, we have the pair of differential equations

$$\frac{dT}{dt} = (a - kD)T - bT^2$$
$$\frac{dD}{dt} = (c - \sigma T)D - dD^2$$

It is thought that $a \approx c$ and $b \approx d$. That is, the two species were essentially alike when considered in isolation. This implies that $a/b \approx c/d$, meaning that in the absence of competition, their population sizes would have been roughly equal. We denote this common logistic equilibrium value by $N$. However, the thylacine was a far more timid creature, while the dingo was bold and aggressive. Consequently, the thylacine was more adversely affected by the presence of the dingo than the other way around.

The key idea is that the two species were almost identical except in how they affected each other. Without dingoes, thylacines filled nearly the same ecological role that dingoes later did. Thus, the only asymmetry lies in their interaction. Clearly, the dingoes have the advantage. Still, could thylacines survive alongside them, perhaps in smaller numbers? In other words, by how much do the dingoes dominate?

Saying the dingoes were less affected by the thylacines means $k$ is large and $\sigma$ is small. To make this comparison independent of units, we will assume that large $k$ means

$$\frac{a}{k} < \frac{c}{d} \approx N.$$

In other words, $a/k$ is smaller than the logistic equilibrium population of the dingoes or thylacines alone, and that small $\sigma$ means

$$N \approx \frac{a}{b} < \frac{c}{\sigma}.$$

That is, $c/\sigma$ is larger than the logistic equilibrium population of the thylacines or dingoes alone.

To find the equilibria, setting $T' = 0$ yields $T = 0$ or $(a - kD) - bT = 0$; setting $D' = 0$ yields $D = 0$ or $(c - \sigma T) - dD = 0$. One can infer that $(0,0)$, $\left(0, \frac{c}{d}\right)$, $\left(\frac{a}{b}, 0\right)$ are equilibrium points. There might be a fourth equilibrium point but only if the two straight lines cross. The $D$- and $T$-axis intercepts are

$$D\text{-axis: } \frac{c}{d}, \frac{a}{k} \quad \text{and} \quad T\text{-axis : } \frac{a}{b}, \frac{c}{\sigma}.$$

The nullclines do not meet in the first quadrant if

$$\frac{a}{k} < \frac{c}{d} \quad \text{and} \quad \frac{a}{b} < \frac{c}{\sigma},$$

Hence, with an introverted thylacine (large impact from dingoes: small $\sigma$) and an extroverted dingo (small impact from thylacines: large $k$), the lines fail to intersect in the first quadrant. Therefore, only the first three equilibria exist under these assumptions.

The Jacobian matrix is

$$\mathbf{J}(T, D) = \begin{bmatrix} a - kD - 2bT & -kT \\ -\sigma D & c - \sigma T - 2dD \end{bmatrix}.$$

At the equilibrium point $\left(0, \frac{c}{d}\right)$, the determinant of the Jacobian matrix is

$$\det(\mathbf{J}) = ck\left(\frac{c}{d} - \frac{a}{k}\right) = ckP,$$

where $P = \frac{c}{d} - \frac{a}{k} > 0$. So, $\det(\mathbf{J}) > 0$. The trace is

$$\operatorname{tr}(\mathbf{J}) = k\left(\frac{a}{k} - \frac{c}{d}\right) - c = -kP - c < 0.$$

Since

$$(\operatorname{tr}(\mathbf{J}))^2 - 4\det(\mathbf{J}) = (kP - c)^2 > 0,$$

the point $\left(0, \frac{c}{d}\right)$ is a nodal sink on the $D$-axis. At the origin, the equilibrium is a nodal source, while the point $\left(\frac{a}{b}, 0\right)$ on the $T$-axis is a saddle. Because the nullclines do not intersect in the first quadrant, there is no interior equilibrium. This means that once dingoes arrive, even in small numbers, the thylacine population is driven toward extinction, while the dingo population stabilises. The model thus predicts total dominance by dingoes and no coexistence.

However, if the two species were less similar, for example, if dingoes had a much higher death rate per capita $d$, then the ratio $c/d$ would decrease. This change allows the nullclines to intersect in the first quadrant, creating a new interior equilibrium. In that case, the origin remains a nodal source, the thylacine equilibrium $\left(\frac{a}{b}, 0\right)$ remains a saddle, the dingo equilibrium $\left(0, \frac{c}{d}\right)$ becomes a saddle, and the new interior point becomes a nodal sink. The two species would then settle at positive, stable populations and coexist.

In summary, species that are too similar cannot coexist as one inevitably eliminates the other. This outcome is known as the principle of competitive exclusion. However, when species differ enough in their ecological roles or weaknesses, stable coexistence becomes possible.

## 5.5 Some Cosmology

There are about 300 billion stars in our galaxy, the Milky Way. This leads many people to say that we cannot be alone in our galaxy: even if intelligent life is rare, there *must* be a few others out there. There is evidence against this, however. Travelling here in person to visit us is too difficult — the distances are too great. But the aliens could build *self-replicating probes* (sometimes called *Von Neumann probes*) to do the exploring for them. These are machines programmed to make copies of themselves as they encounter resources (planets, asteroids etc.) while wandering through space, and then send a message home to report the existence of other civilisations.

It is estimated that by doing this, an alien civilisation (or indeed humanity) could explore the entire galaxy in as little as five million years which is a very short time. Why? Because if we program the probes to have a constant 'birth rate per capita', their population would grow exponentially, as in the Malthusian model. The problem is this: we have not seen any such probes. This suggests that aliens do not exist.

Let $P(t)$ denote the number of correctly programmed probes and let $M(t)$ denote the number of mutants. Since the galaxy has so many stars and planets, there is effectively no logistic limit so we just use the Lotka-Volterra equations but with a twist. Our equations are

$$\frac{dM}{dt} = uPM - D_M M + mP$$
$$\frac{dP}{dt} = B_P P - sMP - mP$$

Here, we are assuming that probes are born at a constant birth rate per capita, and that their death rate per capita is proportional to the number of mutants, while the birth rate per capita of mutants is proportional to the number of probes (their food). The mutants also wear out eventually and die at some constant rate per capita. On top of that, probes are being turned into mutants at a rate $mP$ as we said.

## 5.6 Non-Linear Second-Order ODEs

As we know, non-linear second-order differential equations can be difficult to handle. However, the methods of this chapter can help. Consider the differential equation

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{K}{m}\dot{\theta}.$$

Recall that this is the equation of a damped pendulum in full generality, where we assume that a frictional force is proportional to the speed. Here $K$ is the constant of proportionality and $L\dot{\theta}$ is the angular velocity. Again, we use the following trick — this second-order differential equation is equivalent to a pair of first-order differential equations

$$\dot{\theta} = \psi \quad \text{and} \quad \dot{\psi} = -\frac{g}{L}\sin\theta - \frac{K}{m}\psi$$

where the first equation is the definition of $\psi$. The equilibria are at $(0,0)$ and $(\pi,0)$. Thereafter, find the Jacobian matrices corresponding to these equilibrium points.

We then give another application related to planetary motion. Let $r(t)$ denote the distance from the centre of the Sun to the Earth. It turns out that using simple Physics that the equation governing the motion of a planet going around the Sun is

$$\ddot{r} = -\frac{M}{r^2} + \frac{L^2}{r^3}. \tag{5.2}$$

Here, $M$ denotes the mass of the sun and $L^2$ is a constant that depends on the planet (in fact, on its angular momentum). The second term $L^2/r^3$ has nothing to do with gravity and it is always he same no matter what the law of gravitation may be. On the other hand, the first term comes from the inverse square law of gravitation. Note that we are focusing on the radial motion of the planet, i.e its motion towards and away from the Sun. As such, at this juncture, we ignore the angular motion.

Note that (5.2) is a non-linear second-order differential equation. Converting it into a system of first-order differential equations, we have

$$\dot{r} = R \quad \text{and} \quad \dot{R} = -\frac{M}{r^2} + \frac{L^2}{r^3}.$$

The only equilibrium point is at $\left(\frac{L^2}{M}, 0\right)$ which corresponds to a circular orbit of radius $a = \frac{L^2}{M}$, the only kind for which $r$ is constant. The Earth's orbit is in fact extremely close to being circular, so we are very near to this equilibrium point. One can see that

$$\mathbf{J}(r,R) = \begin{bmatrix} 0 & 1 \\ \frac{2M}{r^3} - \frac{3L^2}{r^4} & 0 \end{bmatrix}$$

which at the equilibrium point is

$$\mathbf{J}\left(\frac{L^2}{M}, 0\right) = \begin{bmatrix} 0 & 1 \\ -\frac{M}{a^3} & 0 \end{bmatrix}.$$

This represents a centre. As such ,the phase diagram is a closed curve, at least near to equilibrium, and $r$ must be a periodic function of time which fluctuates between some minimum and maximum, i.e. Earth is sometimesa little closer and sometimes a ltitle further from the Sun, but $r(t)$ is still bounded above and below so it will not become very small or very large.

Example 5.10 (MA3264 AY25/26 Sem 1 Tutorial 10). Consider a Universe in which the law of gravitation is

$$F = -\frac{M}{r^n} \quad \text{where } n > 3.$$

Prove that circular orbits are unstable for all such $n$.

*Solution.* Consider

$$\dot{r} = R \quad \text{and} \quad \ddot{r} = -\frac{M}{r^n} + \frac{L^2}{r^3}.$$

Equilibrium means that $\dot{r}=0$ and $\dot{R}=0$. So, $\frac{M}{r^n}=\frac{L^2}{r^3}$ which implies $\frac{M}{L^2}=r^{n-3}$. The radius is $r=\left(\frac{M}{L^2}\right)^{\frac{1}{n-3}}$. The Jacobian matrix is

$$\mathbf{J}(r,R)=\begin{pmatrix}0 & 1\\ \frac{nM}{r^{n+1}}-\frac{3L^2}{r^4} & 0\end{pmatrix}.$$

Let $a$ denote the mentioned radius. Then,

$$\mathbf{J}(a,0)=\begin{pmatrix}0 & 1\\ \frac{L^2}{a^4}(n-3) & 0\end{pmatrix}$$

and since $n>3$, this represents a saddle point. As such, circular orbits exist in all of these cases, but in all of them these circular orbits are unstable. $\square$

Example 5.11 (MA3264 AY25/26 Sem 1 Tutorial 10). In 1917, Einstein proposed a model describing the size of the Universe, $R(t)$, governed by

$$\frac{d^2R}{dt^2}+\frac{4\pi G\rho_0}{3R^2}=\frac{\lambda R}{3}\quad\text{where } G,\rho_0,\lambda>0.$$

Find the equilibrium solution and explain why it was a mistake.

*Solution.* Let

$$\dot{R}=S\quad\text{and}\quad\dot{S}=\frac{\lambda R}{3}-\frac{4\pi G\rho_0}{3R^2}.$$

The Jacobian matrix is

$$\mathbf{J}(R,S)=\begin{bmatrix}0 & 1\\ \frac{\lambda}{3}+\frac{8\pi G\rho_0}{3R^3} & 0\end{bmatrix}.$$

The only equilibrium point is $S=0$ and $R=R_0$, where $R_0=\left(\frac{4\pi G\rho_0}{\lambda}\right)^{1/3}$. This means that $R$ is a constant so the size of the Universe never changes, which is what Einstein thought was the case. At this equilibrium, one has

$$\mathbf{J}(R_0,0)=\begin{bmatrix}0 & 1\\ \lambda & 0\end{bmatrix}$$

which is a saddle and therefore unstable. Although Einstein had theoretically succeeded in obtaining a static Universe, in reality he had not — the slightest perturbation of his model would cause the Universe to either collapse or expand without bound. $\square$

CHAPTER 6

# Modelling with Partial Differential Equations

## 6.1 Introduction

**Definition 6.1 (partial differential equation).** A partial differential equation (PDE) is an equation containing an unknown function $u(x,y,\ldots)$ of two or more independent variables $x,y,\ldots$ and its partial derivatives with respect to these variables. We call $u$ the dependent variable.

PDEs allow us to deal with situations where something depends on space as well as time. So far, all the models that we studied so far have only involved variations with time.

We discuss a method to solve PDEs known as the separation of variables. This method can be used to solve PDEs involving two independent variables say $x$ and $y$ that can be separated from each other in the PDE. There are similarities between this method and the technique of separating variables for ODEs in Chapter 1.

We make the following observation. Suppose

$$u(x,y) = X(x)Y(y).$$

Then,

$$\begin{aligned}
u_x &= X'(x)Y(y) \\
u_y &= X(x)Y'(y) \\
u_{xx} &= X''(x)Y(y) \\
u_{yy} &= X(x)Y''(y) \\
u_{xy} &= X'(x)Y'(y)
\end{aligned}$$

Note that each derivative of $u$ remains separated as a product of a function of $x$ and a function of $y$. We can exploit this feature. Consider a PDE of the form

$$u_x = f(x)\,g(y)\,u_y.$$

If a solution of the form $u(x,y) = X(x)\,Y(y)$ exists, then one can deduce that

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y) \cdot \frac{Y'(y)}{Y(y)}.$$

The important observation here is that the LHS is a function of $x$ whereas the RHS is a function of $y$. We conclude that the LHS and RHS both equate to some constant $k$. As such, we obtain the two ODEs as follows:

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad g(y) \cdot \frac{Y'(y)}{Y(y)} = k$$

In fact, it is easy to solve this pair of differential equations!

Example 6.1. Solve $u_x + xu_y = 0$.

*Solution.* Suppose a solution of the form $u(x,y) = X(x)\,Y(y)$ exists. Then, we deduce that

$$\begin{aligned} X'(x)\,Y(y) + xX(x)\,Y'(y) &= 0 \\ \frac{1}{x} \cdot \frac{X'(x)}{X(x)} &= -\frac{Y'(y)}{Y(y)} \end{aligned}$$

As such,

$$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad -\frac{Y'(y)}{Y(y)} = k.$$

This implies that $X(x) = ae^{kx^2/2}$ and $Y = be^{-ky}$ for some constants $aA$ and $b$. As such, the general solution is

$$u(x,y) = X(x)\,Y(y) = ce^{kx^2/2-ky}.$$

Here, $c = ab$ is also a constant. □

## 6.2 The Wave Equation

Consider a flexible string that lies stretched tightly (another word would be 'taut') along the $x$-axis and has its ends fixed at $x = 0$ and $x = \pi$. We pull it along the $u$-axis so that it is stationary and has some specific shape $u = f(x)$ at time $t = 0$. Consequently, $f(0) = 0$ and $f(\pi) = 0$. We can assume that $f(x)$ is continuous and bounded. When we let go of the string, it will move. We assume that the only forces acting are those due to the tension in the string and that the pieces of the string will only move along the $u$-axis.

Now, the $u$-coordinate of any point on the string will become a function of time as well as a function of $x$. So, it becomes a function $u(t,x)$ of both $t$ and $x$. Note that this function satisfies the boundary conditions

$$u(t,0)=0 \quad \text{and} \quad u(t,\pi)=0$$

for all $t$ as the ends are nailed down. Also, the initial condition

$$u(0,x)=f(x) \quad \text{is satisfied.}$$

Also, since the string is initially stationary, then

$$\frac{\partial u}{\partial t}(0,x)=0.$$

We now introduce the wave equation.

**Definition 6.2 (wave equation).** Let $c$ be a fixed non-negative real constant representing the propagation speed of the wave. Then,

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}.$$

Also, $u(t,0)=u(t,\pi)=0$, $u(0,x)=f(x)$ and $u_t$ evaluated at $t=0$ gives 0.

We note that the function

$$u(t,x)=\frac{f(x+ct)+f(x-ct)}{2} \quad \text{is a solution to the wave equation.}$$

More generally, we have D'Alembert's formula (Theorem 6.1)[1]. One should check that the above equation indeed satisfies the wave equation. Moreover, the four conditions should be satisfied.

**Theorem 6.1 (D'Alembert's solution to the wave equation).** The function

$$u(t,x)=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\ d\xi$$

is a solution to the wave equation (Definition 6.2).

*Proof.* By the Fundamental Theorem of Calculus, suppose $G$ is an antiderivative of $g$. Then,

$$\int_{x-ct}^{x+ct} g(\xi)\ d\xi = G(x+ct)-G(x-ct).$$

Hence, D'Alembert's formula becomes

$$u=\frac{f(x+ct)+f(x-ct)}{2}+\frac{G(x+ct)-G(x-ct)}{2c}.$$

So,

$$u_t=c\cdot\frac{f'(x+ct)-f'(x-ct)}{2}+\frac{g(x+ct)+g(x-ct)}{2}$$

[1] Appears in MA3264 AY25/26 Sem 1 Tutorial 10.

so

$$u_{tt} = c^2 \cdot \frac{f''(x+ct) + f''(x-ct)}{2} + c \cdot \frac{g'(x+ct) - g'(x-ct)}{2}.$$

One can then show that

$$u_{xx} = \frac{f''(x+ct) + f''(x-ct)}{2} + \frac{g'(x+ct) - g'(x-ct)}{2c}.$$

Since $u_{tt} = c^2 u_{xx}$, the result follows. $\square$

Initially, $f(x)$ was only defined between $x = 0$ and $x = \pi$, but the interpretation of D'Alembert's solution is that we can extend $f(x)$ to be an odd, periodic function of period $2\pi$.

Consider a function $f(x)$ defined on $[0, \pi]$. We aim to extend $f$ to an odd periodic function with periodic $2\pi$, covering $\mathbb{R}$. First, we define $f$ on $[-\pi, \pi]$ as an odd function, i.e. for any $x \in [-\pi, 0]$, define $f(-x) = -f(x)$. This extension makes $f$ an odd function on $[-\pi, \pi]$. Next, we extend $f$ periodically across the real line, i.e. for any $x$ outside $[-\pi, \pi]$, define

$$f(x) = f(x - 2n\pi) \quad \text{where } n \in \mathbb{Z} \text{ and } x - 2n\pi \in [-\pi, \pi].$$

This creates a periodic function with period $2\pi$.

We then consider the function $f(x - ct)$, where $c > 0$. Note that $f(x-1)$ is the same as $f(x)$, just shifted 1 unit to the right. Similarly, $f(x - ct)$ represents the same shape as $f(x)$ but shifted to the right by $ct$ units. The function $f(x + ct)$ represents $f(x)$ shifted to the left by $ct$, moving in the opposite direction with the same speed.

Geometrically, D'Alembert's solution describes the solution to the wave equation as a combination of two travelling waves, where the term

$$\begin{aligned} &f(x-ct)/2 \quad \text{representsa wave traveling to the right at speed } c \quad \text{and} \\ &f(x+ct)/2 \quad \text{representsa wave traveling to the left at speed } c \end{aligned}$$

Each piece maintains the shape of the original function $f(x)/2$ and moves without distortion.

We can also solve the wave equation using the method of separation of variables. Suppose we wish to solve

$$u_{tt} = c^2 u_{xx}$$

subjected to the following conditions:

$$u(t,0) = u(t,\pi) = 0 \quad u(0,x) = f(x) \quad u_t(0,x) = 0$$

We separate the variables, i.e.

$$u(t,x) = v(x)\,w(t)$$

and obtain

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2}\cdot\frac{w''(t)}{w(t)} = -\lambda.$$

The usual separation argument from before implies $\lambda$ is a constant, so we obtain the following pair of ODEs:

$$v'' + \lambda v = 0 \quad \text{and} \quad w'' + \lambda c^2 w = 0$$

Let us force $v(x)$ to vanish at $x = 0$ and $x = \pi$, so we can set

$$u(t,0) = v(0)\,w(t) = 0 \quad \text{and} \quad u(t,\pi) = v(\pi)\,w(t) = 0.$$

This is somewhat different from the usual scenario of solving second-order ODEs. Normally, we give some information about the function at one point, i.e. we might ask for solutions to the ODE $y'' + \lambda y = 0$ where $y(0)$ and $y'(0)$ are given. However, we are now giving the information from two different points.

If $\lambda < 0$, then $u(0) = 0$ implies that all solutions to the equation $v'' + \lambda v = 0$ are proportional to $\sinh x$, and such a function cannot intersect the $x$-axis twice. As such, $\lambda$ cannot be negative. If $\lambda = 0$, then $v(x)$ is a straight line function which cannot intersect the $x$-axis twice. As such, $\lambda > 0$. We write $\lambda = n^2$ for some $n > 0$. As such,

$$v(x) = C\cos nx + D\sin x \quad \text{for some constants } C \text{ and } D.$$

Since $v(0) = 0$, then $C = 0$ and so $v(x) = D\sin nx$. If we want $v(\pi) = 0$, then it implies $\sin n\pi = 0$. As such, $n \in \mathbb{Z}$ so we also introduce this constraint. Earlier, we mentioned that $n > 0$. Combining both properties, we conclude that $n \in \mathbb{Z}^+$.

Solving the other equation for $w(t)$, we obtain

$$w(t) = A\cos nct + B\sin nct \quad \text{for some constants } A \text{ and } B.$$

We force $w(t)$ to satisfy $w'(0) = 0$ since we want $u_t(0,x) = u(x)\,v'(0) = 0$. So, now $B = 0$ and we are left with $w(t) = A\cos nct$. As such, our complete solution is

$$u(t,x) = b_n \sin nx \cos nct.$$

Here, $b_n$ is an arbitrary constant and again, recall that $n$ is a positive integer. This satisfies three of the four conditions in Definition 6.2. So, the only condition that is not yet satisfied is $u(0,x) = f(x)$.

We recall some concepts from MA2101. Think about the set of all continuous functions on $[0,\pi]$. It is a vector space over $\mathbb{R}$ (an obvious fact). What is a possible basis for it? Well, an example of a basis is given by the set $\{\sin nx : n \in \mathbb{Z}^+\}$. In other words, any continuous function $g$ on $[0,\pi]$ can be expressed as the following:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

This holds for certain real numbers $b_n$. In particular, the formula for $b_n$ is

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\,dx.$$

That is, $2/\pi$ times the integral plays the role of the scalar product here. The series

$$\sum_{n=1}^{\infty} b_n \sin x \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\,dx$$

is known as the Fourier series of $g(x)$. There is an amazing fact that the Fourier series allows us to express any function on this interval as the components — $b_n$! We now return to the problem of solving the wave equation. Recall that we have extended $f(x)$ to be an odd function of period $2\pi$. As such, it has a Fourier sine series, and since $f$ is continuous and has only a finite number of sharp corners, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Now, consider the series

$$\sum_{n=1}^{\infty} b_n \sin nx \cos nct \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

First, observe that if we substitute $t=0$ in this series, then we obtain $f(x)$ expressed as its Fourier sine series. Next, since the wave equation is linear and each term in this series is a solution to the wave equation, then this series is also a solution to the wave equation.

To summarise, the solution to the wave equation is

$$u(t,x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx\right)\sin nx \cos nct \tag{6.1}$$

We have done everything for the interval $[0,\pi]$. For a general interval $[0,L]$ of any length $L$, it is easy to obtain a solution to the modified wave equation. The basis functions are now $\sin(n\pi x/L)$ which are periodic with period $2L$ instead of $2\pi$ like before. The Fourier series formulae are

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad b_n = \frac{2}{L}\int_0^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

$f$ will now be a function that vanishes at 0 and $L$!

Example 6.2 (MA3264 AY25/26 Sem 1 Tutorial 10). Solve the wave equation

$$c^2 y_{xx} = y_{tt}$$

on $[0,\pi]$, with boundary conditions $y(t,0) = y(t,\pi) = 0$, initial velocity $y_t(0,x) = 0$, and initial displacement $y(0,x) = x$ for $0 \le x < \pi$, $y(0,\pi) = 0$.

*Solution.* We write the wave equation as $u_{tt} = c^2 u_{xx}$. Assume a solution of the form

$$u(x,t) = X(x)T(t).$$

Then,

$$u_{tt} = X(x)T''(T) \quad \text{and} \quad u_{xx} = X''(x)T(t).$$

So,

$$X(x)T''(T) = c^2 X''(x)T(t).$$

We have

$$\frac{X''(x)}{X(x)} = -\lambda \quad \text{and} \quad \frac{T''(t)}{c^2 T(t)} = -\lambda.$$

For the first differential equation, we obtain $X''(x) + \lambda X(x) = 0$, which has characteristic equation $m^2 + \lambda = 0$. We require $X(0) = 0$ and $X(\pi) = 0$. Hence, $X(x)$ must be a sine series solution. Suppose the eigenvalues are $\lambda = n^2$ for $n = 1, 2, 3, \ldots$ with eigenfunctions $X_n(x) = \sin(nx)$.

The time differential equation $T'' + c^2 n^2 T = 0$ yields

$$T_n(t) = A_n \cos(cnt) + B_n \sin(cnt). \tag{6.2}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(cnt) + B_n \sin(cnt)\right] \sin(nx).$$

Since $u_t(x,0) = 0$, then $B_n = 0$ for all $n \in \mathbb{N}$. It remains to determine the coefficient $A_n$. We have $u(x,0) = x$. We just need to extend $f(x) = x$ to an odd function on $(-\pi, \pi)$ and then turn that into a periodic function of period $2\pi$. By considering (6.2),

$$\sum_{n=1}^{\infty} A_n \sin(nx) = x.$$

We have

$$A_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx = \frac{2(-1)^{n+1}}{n}$$

so we conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cos(cnt) \sin(nx).$$

□

Example 6.3. Consider the wave equation with the following initial conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1,\ t > 0; \\ u(0,t) = u(1,t) = 0 & t \geq 0; \\ u(x,0) = \sin(\pi x) + 0.3 \sin(2\pi x) & 0 \leq x \leq 1; \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & 0 \leq x \leq 1 \end{cases}$$

Recall from (6.1) that the solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi ct) \quad \text{where} \quad a_n = 2\int_0^1 u_0 \sin(n\pi x)\, dx.$$

Here,

$$u_0(x) = \sin(\pi x) + 0.3\sin(2\pi x).$$

Hence, the general solution is

$$u(x,t) = \sin(\pi x)\cos(\pi ct) + 0.3\sin(2\pi x)\cos(2\pi ct).$$

Example 6.4 (MA3264 AY25/26 Sem 1 Tutorial 10). If a flexible string moves in a fluid with friction proportional to velocity, its motion is governed by the damped wave equation

$$u_{tt} = c^2 u_{xx} - bu_t,$$

with $b > 0$ and $u(0,t) = u(\pi,t) = 0$. Show how to separate variables, and explain why the resulting ODEs lead to damped harmonic oscillator behaviour.

*Solution.* We seek a solution

$$u(x,t) = X(x)T(t).$$

So,

$$\frac{T''}{c^2 T} + \frac{b}{c^2}\frac{T'}{T} = \frac{X''}{X} = \alpha$$

where $\alpha$ is a constant. Thus, we obtain the pair of differential equations

$$X'' = \alpha X \quad \text{and} \quad T'' + bT' - c^2\alpha T = 0.$$

Since $u(0,t) = 0$ and $u(\pi,t) = 0$, we have $X(0) = 0$ and $X(\pi) = 0$. We must have $\alpha < 0$, otherwise $X$ is exponential or linear and the only function satisfying both Dirichlet conditions is the trivial one. Write $\alpha = -\lambda$, where $\lambda > 0$. Then,

$$X'' + \lambda X = 0 \quad \text{with } X(0) = X(\pi) = 0.$$

We have the eigenvalues $\lambda_n = n^2$ and corresponding eigenfunctions $X_n(x) = \sin(nx)$ for $n = 1, 2, \ldots$. For the temporal part,

$$T_n'' + bT_n' + c^2\lambda_n T_n = 0 \quad \text{so} \quad T_n'' + bT_n' + (cn)^2 T_n = 0.$$

This has characteristic equation $r^2 + br + (cn)^2 = 0$. It has roots

$$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c^2n^2}.$$

This is precisely the damped harmonic oscillator. Each mode $X_n(x)$ evolves in time as a damped oscillator with natural frequency and linear damping $b$. Thus the separated solutions are

$$u_n(x,t) = T_n(t)\sin(nx)$$

and superpositions $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx)$ (with coefficients fixed by initial data) exhibit oscillations that decay like $e^{-bt/2}$ in the underdamped case, or purely decaying exponentials otherwise — precisely the behaviour of damped harmonic oscillators. $\square$

We now explore an application of the wave equation to tsunamis. A tsunami is a wave in the ocean, usually generated by an undersea earthquake. Out at sea, it moves extremely fast and is quite small in height (approximately 1 metre). These statistics are both bad because they make early warning extremely difficult — there are many differently-shaped one-metre tall waves out in the open ocean. One of the most important of all mathematical modelling problems is to understand the characteristic shape of a tsunami when it is still far away from shore — if one can recognise the shape, then one might be able to warn people that a tsunami is coming. One can model a surface wave in shallow water by means of the Korteweg-de Vries equation, which is a partial differential equation defined as follows:

$$\frac{\partial \eta}{\partial t}+\sqrt{gh}\frac{\partial \eta}{\partial x}+\frac{3}{2}\sqrt{\frac{g}{h}}\eta\frac{\partial \eta}{\partial x}+\frac{1}{6}h^2\sqrt{gh}\frac{\partial^3 \eta}{\partial x^3}=0, \tag{6.3}$$

where $\eta$ is the elevation above the mean sea level, $h$ is the depth of the sea and $g$ is the acceleration due to gravity. We see that the Korteweg-de Vries equation (6.3) is a third-order non-linear partial differential equation. It is important to note that this equation only works for waves in shallow water. One might think that the ocean is not exactly shallow as the typical depths far away from land are around 3-4 kilometres, but to a tsunami, it is actually shallow becasue the wavelengths of a tsunami wave can stretch around 200 km!

The point is, if we can solve the Korteweg-de Vries equation, the shape of the wave we find will be the shape of a tsunami and not of any other kind of wave. Hence, the shape we find will allow us to identify tsunamis. In other words, we can distinguish them from all of the other waves in the ocean. This model is specifically designed to analyse tsunamis while they are still far from shore. Near the coastline, the behavior of a tsunami becomes significantly more complex — it slows down and grows much taller, making accurate modeling more difficult. Fortunately, our primary goal is to detect and understand tsunamis in the deep ocean, where they are still moving quickly and can be identified with greater precision.

We now look for a wave solution. From what we have seen, that means we wish to find a solution of the form $E(x-ct)$, a wave of fixed shape given by the form of the function $E$ moving to the right at speed $c$. Substituting this into the Korteweg-de Vries equation yields

$$\left(-c+\sqrt{gh}\right)E'+\frac{3}{2}\sqrt{\frac{g}{h}}EE'+\frac{1}{6}h^2\sqrt{gh}E'''=0.$$

Letting $A,B,C$ be constants, we have

$$-2AE'+6BEE'+2CE'''=0. \tag{6.4}$$

Since the derivative of $E^2$ is $2EE'$, (6.4) becomes

$$-2AE+3BE^2+2CE''=0.$$

So,

$$-AE^2+BE^3+C\left(E'\right)^2=k$$

where $k$ is a constant of integration. We have obtained a separable first order ordinary differential equation! A typical solution of this equation involves $\operatorname{sech}^2$, where sech denotes hyperbolic secant. As such, our model predicts that a tsunami should move at a steady speed with that exact shape, which of course one can easily graph. This model is simple, but still informative. The linear wave equation allows waves of any shape, meaning it cannot predict or constrain form. In contrast, the non-linear Korteweg-de Vries equation does determine shape. When we searched for a wave-like solution, the form emerged directly from the equation. This gives hope: a better non-linear model might predict tsunami shapes, helping us distinguish them from other waves and enabling early detection and warnings.

Example 6.5 (MA3264 AY25/26 Sem 1 Tutorial 11). We saw that the shape of a wave solution of the Korteweg-de Vries equation is found by solving the equation

$$-AE^2+BE^3+C\left(E'\right)^2=K$$

for the function $E(p)$. Here $p$ is used as the variable. Consider the special case $A=B=C=1$ and $K=0$.

**(a)** Confirm by direct substitution that, if $E(0)=1$, then a solution is

$$E(p)=\operatorname{sech}^2\left(\frac{p}{2}\right).$$

So basically $\operatorname{sech}^2(p/2)$ is the simplest solution of the KdV equation.

**(b)** There are many other solutions to the KdV equation. Show that, with these parameters, $E(0)>1$ is not possible. So some initial conditions are impossible; the equation dictates the kind of initial conditions it will accept. What is the solution if $E(0)=a$, where $0<a<1$?

*Solution.*

**(a)** We first consider the case where $A=B=C=1$ and $K=0$, for which we obtain

$$-E^2+E^3+\left(E'\right)^2=0.$$

One can substitute $E(p)=\operatorname{sech}^2\left(\frac{p}{2}\right)$ into the differential equation to verify that it indeed satisfies it.

**(b)** We have

$$\left(\frac{dE}{dp}\right)^2=E^2(1-E).$$

So,

$$\frac{dE}{dp}=\pm E\sqrt{1-E}.$$

For real $E(p)$, we must have $1-E\geq 0$ so $E\leq 1$. As such, $E(0)>1$ is not possible. One can then use the substitution $E=\sin^2\theta$ to solve the differential equation. We omit the details. $\square$

Example 6.6 (MA3264 AY25/26 Sem 1 Tutorial 11). Previously, we stressed that the Korteweg-de Vries equation dictates the shape of its wave solutions. What happens if you try to get that to work for the wave equation? That is, take the wave equation and look for solutions of the form $E(x-Ct)$, where this constant $C$ need not be the same as the $c$ in the wave equation (Definition 6.2).

*Solution.* Suppose
$$u(x,t) = E(x-Ct).$$
Then,
$$u_t = -CE(x-Ct) \quad \text{so} \quad u_{tt} = C^2E(x-Ct)$$
and
$$u_{xx} = E(x-Ct).$$
It is clear that $u_{tt} = C^2u_{xx}$, so □

## 6.3 The Heat Equation

Consider the temperature in a long thin bar or wire of constant cross-section and homogeneous material which is oriented along the $x$-axis and is perfectly insulated laterally, so that heat only flows in the $x$-direction. Then the temperature $u$ depends only on $x$ and $t$ and is given by the one-dimensional heat equation.

Definition 6.3 (heat equation). The heat equation states that
$$u_t = c^2u_{xx},$$
where $c^2$ is a positive constant called the thermal diffusivity (sometimes this is denoted by $\kappa$). It measures how quickly heat moves through the bar and depends on what it is made of.

Let us assume that the ends $x=0$ and $x=L$ of the bar are kept at temperature zero, so that we have the following boundary conditions:
$$u(0,t) = 0 \quad u(L,t) = 0 \quad \text{for all } t,$$
and the initial temperature of the bar is $f(x)$, so that we have the initial condition
$$u(x,0) = f(x).$$
Here, we will assume that when $f(x)$ is extended to be an odd function, it equals its Fourier sine series everywhere. Remember that this can happen, even if $f(x)$ is discontinuous at some points. Notice that unlike the wave equation, which needs four pieces of data, here we only need three, which matches the fact that the heat equation only involves a total of three derivatives (two in the spatial direction, but only one in the time direction).

The heat equation is particularly useful in modeling for the following reason. Think of an ordinary function, $g(x)$. We can think of its second derivative $g''(x)$ as a measure of the extent to which its graph is not a straight line (recall that the second derivative is zero everywhere if and only if $g(x)$ is a linear function). We say that $g''(x)$ measures the curvature of the graph.

The heat equation says that the second spatial derivative of $u$ is equal to its time derivative. So as time goes by, if the graph of $u$ as a function of $x$ is concave up, then $u$ will increase; whereas if the graph is concave down, then it tends to decrease. The effect in both cases is to reduce the curvature. So we can picture the equation as something that, given an initial shape described by $f(x)$, tries to "straighten it out." And of course, that is how we expect heat to behave, i.e. heat flows from a hotter region to a cooler region, trying to even out its distribution.

It turns out that the solution to the one-dimensional heat equation looks like this.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2c^2}{L^2}t\right) \tag{6.5}$$

where the $b_n$ are just the Fourier sine coefficients of $f(x)$.

Notice that we get exponentials here instead of sines. That is because the separated equation for the function of $t$ is first-order (for obvious reasons), and as we know, first-order ordinary differential equations tend to have exponential solutions. Because of this, the solutions to the heat equation depend on the direction of time. This means that this PDE is useful for modeling situations involving irreversible time evolution.

Example 6.7 (MA3264 AY25/26 Sem 1 Tutorial 10). Solve the heat equation

$$y_t = c^2 y_{xx}$$

on $[0,\pi]$, with $y(0,t) = y(\pi,t) = 0$ and $y(x,0) = x$ for $0 \le x < \pi$, $y(\pi,0) = 0$.

*Solution.* Recall from (6.5) that the solution to the one-dimensional heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2c^2}{L^2}t\right).$$

One can solve for the Fourier coefficient $b_n$. Take $L = \pi$. Then,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

and you should be able to deduce that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx)\exp\left(-n^2c^2t\right).$$

□

Example 6.8 (MA3264 AY25/26 Sem 1 Tutorial 11). In the notes, we showed that the heat equation

$$u_t = c^2 u_{xx},$$

with

$$u(0,t) = u(L,t) = 0 \quad \text{and} \quad u(x,0) = f(x)$$

has solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{\pi^2 n^2 c^2}{L^2} t\right),$$

where the $b_n$ are the Fourier sine coefficients of $f(x)$. But what happens if the temperature is different at the right end of the bar from the temperature at the left end? For example, what do we do if $u(0,t) = 0$, $u(x,0) = f(x)$, but $u(L,t) = T$, for some constant $T$, meaning that the left side has temperature zero but the right side does not?

Here we use the following trick: consider the function

$$u^*(x,t) \equiv u(x,t) - \frac{Tx}{L}.$$

Solve for it, and then obtain $u(x,t)$ as $u(x,t) = u^*(x,t) + \frac{Tx}{L}$. Check that $u^*(x,t)$ satisfies the heat equation and $u^*(0,t) = u^*(L,t) = 0$; but be careful, $u^*(x,0) \neq f(x)$.

*Solution.* We have

$$u^*(0,t) = u(0,t) = 0 \quad \text{and} \quad u^*(L,t) = u(L,t) - T = 0.$$

Also,

$$u^*(x,0) = u(x,0) - \frac{Tx}{L} \neq f(x).$$

One can check that $u^*(x,t)$ also satisfies the heat equation. So,

$$u^*(x,t) = \sum_{n=1}^{\infty} b_n^* \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{\pi^2 n^2 c^2}{L^2} t\right),$$

where

$$b_n^* = \frac{2}{L} \int_0^L \left(f(x) - \frac{Tx}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

□

## 6.4 Fisher's Equation

Life on dry land took a long time to evolve: animals and plants had lived in the sea for hundreds of millions of years before that happened, roughly 450 million years ago. Of course, it must have started along the sea shore, that is, along a line. There must have been some kind of marine plant growing along the shore line; a mutation occurred (helped by the extreme exposure to sunlight) which made one of them, at some particular time and place, better able to tolerate drying out. The descendants of that individual had a tremendous advantage over the non-mutated neighbours because sometimes there is a succession of exceptionally low tides which leave the plants dry for a long time. So they would have outcompeted their neighbours, and the mutation would have spread along the shoreline like a wave. Eventually, the result would be a plant that could survive out of the water full-time.

The process of spreading along the shoreline is clearly irreversible, so we need an equation like the heat equation, not the wave equation: we need a heat equation with a wave-like solution! On the other hand, we do not want the effect to go away, like the temperature going down as heat dissipates. What we need is a combination of the Heat Equation with our model of the spread of a rumour. In 1937, Ronald Fisher proposed the following equation to model this situation:

$$u_t = \alpha u_{xx} + \beta u(1-u)$$

where $u(x,t)$ is the fraction of the plants at any given place and time which have mutated (so $1-u(x,t)$ is the fraction which haven't). This is indeed a combination of the heat equation with the rumour equation! The constant $\alpha$ tells you how quickly the mutation tends to spread in space, while $\beta$ measures how quickly it grows in time at a specific point in space (they have different units, of course).

This is a non-linear partial differential equation, and finding all of its solutions is very difficult. But it is important because it has many other applications, for example to the theory of how flames move and to the theory of how nuclear reactors work. To solve this equation, we specify some initial function $f(x) = u(x,0)$ and then try to evolve it forward in time. A good model for $f(x)$ would be a delta function.

We seek a wave solution of the form

$$u(x,t) = U(x-ct)$$

where $U(s)$, $s = x-ct$, describes the wave moving to the right at constant speed $c$. Substituting this into Fisher's equation gives

$$\alpha U'' + cU' + \beta U - \beta U^2 = 0.$$

This ODE has two equilibria: $(U,U') = (0,0)$ and $(U,U') = (1,0)$. The Jacobian at these equilibria determines their stability. We have

$$\mathbf{J}(U,U') = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{\alpha}(2U-1) & -\frac{c}{\alpha} \end{bmatrix}.$$

We have

$$\mathbf{J}(1,0) = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{\alpha} & -\frac{c}{\alpha} \end{bmatrix}$$

which is a saddle. Next, we have

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ -\frac{\beta}{\alpha} & -\frac{c}{\alpha} \end{bmatrix}$$

which is a spiral sink if $c < 2\sqrt{\alpha\beta}$. Since a spiral sink at the origin will inevitably lead to negative values of $U$, which does not make sense, we reject this opinion and insist that $c \geq 2\sqrt{\alpha\beta}$. So, the model predicts that the wave cannot move more slowly than this and this gives us a possible way of verifying it.

In the case $\mathbf{J}(0,0)$, the origin is a nodal sink so $U$ does not have to be negative. From the phase diagram, there is a unique trajectory connecting the two equilibria, flowing out of the saddle and into the nodal sink. Note that trajectories leaving the saddle in the direction of increasing $U$ are invalid, since clearly $U \leq 1$. Also recall that time in the phase diagram is $s$, not $t$. As $s \to -\infty$, we are near the saddle $(0,1)$; as $s \to +\infty$, we approach the node $(0,0)$. Translating back to $t$, this means that as $t \to -\infty$, we begin at the node $(0,0)$ — a flat solution at zero and as $t \to +\infty$, we approach the saddle $(0,1)$—a flat solution at one. In short, the wave transforms a flat zero state into a flat one state over time. Exactly as desired: the mutation spreads outward from its origin and eventually dominates the entire shoreline.

To see this more precisely, fix $t = t_0$ and consider large negative $x$. Then $s = x - ct \ll 0$, so we're near the saddle $(0,1)$ — the solution is nearly 1 far to the left. Likewise, for large $x \gg 0$, $s \gg 0$, and the solution is near the node $(0,0)$ — close to zero. In between, the wave smoothly connects these states. As time progresses, this transition zone moves rightward, converting more of the graph into the flat 1 state.

A similar process happens with the left-moving wave described by $x + ct$. Eventually, both waves consume the entire domain, leaving the shoreline fully overtaken by the mutation.

Example 6.9 (MA3264 AY25/26 Sem 1 Tutorial 11). The Fisher-Kolmogorov-Petrovsky-Piscunov (FKPP) equation generalizes Fisher's equation: it takes the form

$$u_t = \alpha u_{xx} + \beta u(1 - u^p),$$

where $p$ is a positive number.

**(a)** Repeat the discussion in the lectures, and find the minimum possible speed of a wave solution of the FKPP equation. How do the solutions of this equation differ from those of Fisher's? To find out, study the saddle equilibrium in more detail (show that the node is exactly the same as Fisher's).

**(b)** Set $\alpha = \beta = 1$, $c = 3$, and find the saddle eigenvector of the form $(1, \lambda)$ with $\lambda > 0$ and show that $\lambda$ is an increasing function of $p$.

*Solution.*

**(a)** We seek a solution of the form $u(x,t) = u(x - ct)$, where $c$ denotes the speed of the wave. Then,

$$u_t(x,t) = -cu'(x - ct) \quad \text{and} \quad u_{xx}(x,t) = u''(x - ct).$$

As such,

$$-cu' = \alpha u'' + \beta u - \beta u^{p+1}.$$

Let $v = u'$ so

$$v' = -\frac{c}{\alpha}v - \frac{\beta}{\alpha}u + \frac{\beta}{\alpha}u^{p+1}.$$

Setting $u' = 0$, we have $v = 0$; setting $v' = 0$, we have $-cv - \beta u + \beta u^{p+1} = 0$. So, one equilibrium point is $(u,v) = (0,0)$. The other equilibrium point is $(u,v) = (1,0)$. The Jacobian matrix is

$$\mathbf{J}(u,v) = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{\alpha}\left((1+p)u^p - 1\right) & -\frac{c}{\alpha} \end{bmatrix}. \tag{6.6}$$

Note that $(0,0)$ is a saddle point;

$$\mathbf{J}(1,0) = \begin{bmatrix} 0 & 1 \\ -\frac{\beta}{\alpha} & -\frac{c}{\alpha} \end{bmatrix}.$$

To obtain a spiral sink, we need complex eigenvalues with negative real part. The eigenvalues of the Jacobian matrix $\mathbf{J}(1,0)$ are

$$\frac{-c \pm \sqrt{c^2 - 4\alpha\beta}}{2\alpha}$$

so $c^2 - 4\alpha\beta < 0$ leads to a spiral sink. As such, to avoid having a spiral sink, we need $c^2 - 4\alpha\beta \geq 0$ so $c \geq 2\sqrt{\alpha\beta}$.

**(b)** By (6.6), we have

$$\mathbf{J}(0,0) = \begin{bmatrix} 0 & 1 \\ (1+p)u^p - 1 & -3 \end{bmatrix}.$$

Let $a = (1+p)u^p - 1$. Then the characteristic equation is $\lambda^2 + 3\lambda - a = 0$. So,

$$\lambda = \frac{-3 \pm \sqrt{5 + 4(1+p)u^p}}{2}.$$

Since we wish to work with the eigenvector $(1,\lambda)$, the eigenvalue is

$$\lambda = \frac{-3 + \sqrt{5 + 4(1+p)u^p}}{2}.$$

which is an increasing function of $p$. $\square$