General Vector Spaces and Isomorphisms

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Throughout the whole of MA2001 Linear Algebra I, we are only interested in the Euclidean n-space, which is denoted by $\mathbb{R}^n$. Students would know that this is a finite-dimensional vector space over $\mathbb{R}$ and in fact, it is of dimension n. Let F be an arbitrary field (you do not really need to care what this is but some examples include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the finite field of p elements $\mathbb{F}_p$ where p is a prime).

To start off, recall that $\mathbb{R}^3$ is a 3-dimensional vector space over $\mathbb{R}$. For any arbitrary vector space V over a field F, let $\dim_F V$ denote the dimension of V over F. Explicitly, we have $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$ equipped with the standard basis vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$. Since these standard basis vectors are linearly independent over $\mathbb{R}^3$ and they span $\mathbb{R}^3$, it follows that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$. Any student doing MA2001 should be well-versed with this argument.

Having said that, note that $\dim_{\mathbb{Q}}(\mathbb{R}^3)$ is infinite. In this situation, the scalars are restricted, so we cannot generate every vector in $\mathbb{R}^3$ from only three directions using rational coefficients. For example, the vector $(\pi, 0, 0)$ cannot be written as a rational multiple of $(1,0,0)$ since π is irrational. This means that we would need to include $(\pi, 0, 0)$ in the spanning set if we wanted to cover that direction with rational coefficients. Keep in mind we have other irrational numbers like $\sqrt{2}$. So, $(\sqrt{2}, 0, 0)$ is still not covered. As such, we would need to add it, and so on. This suggests we need infinitely many basis elements.

Consider the following vector spaces over $\mathbb{R}$, which are not taught in the MA2001 curriculum but you might encounter them in other Mathematics courses.

- Fix positive integers m and n. Let

  $$\mathcal{M}_{m\times n}(\mathbb{R}) = \{\text{set of } m \times n \text{ matrices with real entries}\}$$

  which has the standard basis $\{\mathbf{E}_{ij} : 1 \le i \le m, 1 \le j \le n\}$, where $\mathbf{E}_{ij}$ is the matrix with 1 in the (i,j)-entry and 0 elsewhere. For example, $\mathcal{M}_{2\times 2}(\mathbb{R})$ has the standard basis

  $$\{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{21}, \mathbf{E}_{22}\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

  Clearly, $\dim_{\mathbb{R}}(\mathcal{M}_{m\times n}(\mathbb{R})) = mn$ since any basis for $\mathcal{M}_{m\times n}(\mathbb{R})$ contains mn elements (matrices to be precise).

- Fix a non-negative integer n. Let

  $$\begin{aligned} \mathcal{P}_n(\mathbb{R}) &= \{\text{set of polynomials of degree at most } n \text{ with real coefficients}\} \\ &= \left\{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_0, a_1, a_2, \ldots, a_n \in \mathbb{R} \right\} \end{aligned}$$

  Note that polynomials of degree 0 are constant polynomials, polynomials of degree 1 are linear polynomials, polynomials of degree 2 are quadratic polynomials etc. We see that $\{1, x, x^2, \ldots, x^n\}$ is linearly independent over $\mathcal{P}_n(\mathbb{R})$ and it is also a spanning set for $\mathcal{P}_n(\mathbb{R})$. Hence, we say that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{P}_n(\mathbb{R})$ (in fact, it is the standard basis). As such, we infer that $\dim_{\mathbb{R}}(\mathcal{P}_n(\mathbb{R})) = n + 1$ since there are $n + 1$ elements in the set $\{1, x, x^2, \ldots, x^n\}$.

It turns out that if V and W are finite-dimensional vector spaces over $\mathbb{R}$, if these vector spaces have the same dimension (i.e. $\dim(V) = \dim(W)$), then V and W *have a similar structure.* Mathematically, we say that these vector spaces are isomorphic. Let us see what we really mean by this fact. Recall that $\mathbb{R}^n$ is finite-dimensional and $\dim_{\mathbb{R}}(\mathbb{R}^n) = n$, and $\mathcal{P}_{n-1}(\mathbb{R})$ is also finite-dimensional and $\dim_{\mathbb{R}}(\mathcal{P}_{n-1}(\mathbb{R})) = n$. Take any vector $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a column vector, and say we wish to map it to some polynomial in $\mathcal{P}_{n-1}(\mathbb{R})$. Explicitly, we have

$$T : \mathbb{R}^n \to \mathcal{P}_{n-1}(\mathbb{R}) \quad \text{where} \quad \mathbf{x} \mapsto a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}.$$

One can prove that T is a bijective linear map (students doing MA1100 Basic Discrete Mathematics should indeed verify this), which in turn shows that T is an isomorphism.

So far, we have only discussed one example of an infinite-dimensional vector space — $\mathbb{R}^3$ over $\mathbb{Q}$. Some might think that this is a boring example, so we present some other interesting ones. You would generally encounter such vector spaces in higher-level Mathematics courses like MA3209 Metric and Topological Spaces, MA4262 Measure and Integration, MA4211 Functional Analysis etc.

Here are some examples of infinite-dimensional vector spaces you can keep in mind. These vector spaces are infinite-dimensional because no finite list of vectors can span them. In other words, you cannot find a finite basis.

- For example, consider the function space $\mathcal{C}([0,1])$, which denotes the space of all continuous real-valued functions on the interval $[0,1]$ with vector addition and scalar multiplication defined pointwise.
- Sequence spaces are somewhat more interesting. Define ℓ^2 to be the space of all square-summable sequences. That is,

  $$\ell^2 = \left\{ (a_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

  For example, the famous Euler 2-series converges

  $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

  so $\left(\frac{1}{n^2}\right)_{n \in \mathbb{N}} \in \ell^2$. We also have the ℓ^∞ space which is the space of all bounded sequences, and ℓ^1 which is the space of absolutely summable sequences.
- In Functional Analysis, we are concerned with Banach spaces (a normed space) and Hilbert spaces (a space that is equipped with an inner product). It is a known fact that every Hilbert space is also a Banach space. Consider $L^2([0,1])$ which denotes the space of square-integrable functions on $[0,1]$. That is,

  $$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{R} : f \text{ measurable and } \int_0^1 |f(x)|^2 \, dx < \infty \right\}.$$

  This space is also infinite-dimensional!

A well-known fact, as mentioned is that if V and W be finite-dimensional vector spaces over $\mathbb{R}$ and $\dim(V) = \dim(W)$, then V is isomorphic to W. We write $V \cong W$ to denote 'V is isomorphic to W'. More generally, this statement holds for any arbitrary field F.

Let V and W be finite-dimensional vector spaces over a field F. Define $\text{Hom}_F(V, W)$ to be the set of linear transformations from V to W. This is also denoted by $\mathcal{L}(V, W)$. It is a known fact that

$$\text{Hom}_F(V, W) \text{ is a finite-dimensional vector space} \quad \text{with} \quad \text{dimension } mn.$$

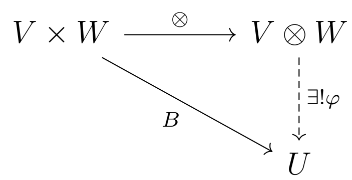
Next, a tensor product of V and W is a vector space $V \otimes W$ over F equipped with a bilinear map

$$\otimes : V \times W \to V \otimes W \quad \text{where} \quad (\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \otimes \mathbf{w},$$

with the following universal property: for every vector space U and every bilinear map $B : V \times W \to U$, there exists a unique linear transformation $\varphi : V \otimes W \to U$ such that

$$B(\mathbf{v}, \mathbf{w}) = \varphi(\mathbf{v} \otimes \mathbf{w}) \quad \text{for all } \mathbf{v} \in V, \mathbf{w} \in W.$$

Pictorally, the following diagram commutes:



Again, it is a known fact that

$$V \otimes W \text{ is a finite-dimensional vector space} \quad \text{with} \quad \text{dimension } mn.$$

As vector spaces, $V \otimes W$ and $\text{Hom}_F(V, W)$ are isomorphic. Having said that, there is a canonical basis-free isomorphism

$$\text{Hom}_F(V, W) = V^* \otimes_F W,$$

where V^* is the dual space of V. Here, we define

$$V^* = \text{Hom}_F(V, F).$$

So, getting the isomorphism $\text{Hom}_F(V, W) \cong V \otimes_F W$ requires an extra, non-canonical choice. We discuss the canonical isomorphism

$$\text{Hom}_F(V, W) \cong V^* \otimes W.$$

Define a bilinear map

$$b : V^* \times W \to \text{Hom}_F(V, W) \quad \text{where} \quad b(f, \mathbf{w}) = f(\mathbf{v})\,\mathbf{w}.$$

In other words,

$$b : \text{Hom}_F(V, F) \times W \to \text{Hom}_F(V, W) \quad \text{where} \quad b(f, \mathbf{w}) = f(\mathbf{v})\,\mathbf{w}.$$

This means that given a linear functional $f \in \text{Hom}_F(V, F)$ and a vector $\mathbf{w} \in W$, we can construct a linear transformation $T : V \to W$ by the rule $T(\mathbf{v}) = f(\mathbf{v})\,\mathbf{w}$. By the universal property of the tensor product, there exists a unique linear transformation

$$\varphi : V^* \otimes W \to \text{Hom}_F(V, W) \quad \text{where} \quad \varphi(f \otimes w)(v) = f(v)\,w.$$

So, the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_F(V,F) \times W & \xrightarrow{\otimes} & \operatorname{Hom}_F(V,F) \otimes W \\ & \searrow^{b} & \Big\downarrow{\scriptstyle \exists!\varphi} \\ & & \operatorname{Hom}_F(V,W) \end{array}$$

To see why φ is an isomorphism, take a basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of V with dual basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^i\} \subseteq V^*$ and a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of W. For a sanity check, $\dim(V) = m$ and $\dim(W) = n$. Then, for each $1 \leq i \leq n$ and $1 \leq j \leq m$, we have

$$\varphi\left(\mathbf{e}^i \otimes \mathbf{w}_j\right) \in \operatorname{Hom}_F(V, W).$$

Note that this is the rank-one map sending $\mathbf{e}_i \mapsto \mathbf{w}_j$ and $\mathbf{e}_k \mapsto 0$ for $k \neq i$. These mn maps form the standard basis of $\operatorname{Hom}_F(V, W) \cong \mathcal{M}_{m \times n}(F)$. Hence ϕ is bijective since it sends a basis to a basis. So,

$$\operatorname{Hom}_F(V, W) \cong V^* \otimes W \text{ naturally.}$$