Dirichlet Character Tables

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A Dirichlet character modulo k, denoted by χ, is an arithmetic function $\chi : \mathbb{N} \to \mathbb{C}$ satisfying the following:

(i) $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$ (i.e. χ is completely multiplicative)

(ii) $|\chi(n)| = 1$ if $\gcd(n, k) = 1$ and 0 otherwise

(iii) $\chi(n + km) = \chi(n)$ for all $m, n \in \mathbb{N}$

We give some examples of constructing Dirichlet character tables. We first do for modulo 8. Modulo 8, the reduced system is $(\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\}$ which is a group of order 4. Moreover, every element is of order 2 so we have the following isomorphism: $(\mathbb{Z}/8\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. As such, there are exactly 4 Dirichlet characters modulo 8.

We take 3 and 5 as independent generators. Since χ is multiplicative, then $\chi(15) = \chi(3)\chi(5)$. By periodicity, $\chi(15) = \chi(7)$ so $\chi(3)\chi(5) = \chi(7)$. So, a possibility is $\chi(3) = 1$ and $\chi(5) = -1$, which would imply $\chi(7) = -1$. As such, we have the following Dirichlet character table. Note that ordering of the rows is arbitrary except the first. Other tables may have the rows in a different order.

n	1	3	5	7
χ_0	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

We then construct the Dirichlet table modulo 20. Again, we find the reduced system, which is

$$(\mathbb{Z}/20\mathbb{Z})^\times = \{1, 3, 7, 9, 11, 13, 17, 19\}.$$

Since $20 = 4 \cdot 5$, by the Chinese remainder theorem,

$$(\mathbb{Z}/20\mathbb{Z})^\times \cong (\mathbb{Z}/4\mathbb{Z})^\times (\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

For example, $11 \mapsto 1$ and $3 \mapsto 9 \mapsto 7 \mapsto 1$. The first map tells us that $\chi(11) = \pm 1$ and the second map tells us that $\chi(3) \in \{1, -1, i, -i\}$. Suppose $\chi(11) = -1$ and $\chi(3) = -i$. Then, $\chi(9) = -1$ and so $\chi(7) = i$. These correspond to the entries in red as in χ_5. As mentioned, the ordering of the rows is arbitrary except the first!

n	1	3	7	9	11	13	17	19
χ_0	1	1	1	1	1	1	1	1
χ_1	1	i	$-i$	-1	1	i	$-i$	-1
χ_2	1	-1	-1	1	1	-1	-1	1
χ_3	1	$-i$	i	-1	1	$-i$	i	-1
χ_4	1	1	1	1	-1	-1	-1	-1
χ_5	1	i	$-i$	-1	-1	$-i$	i	1
χ_6	1	-1	-1	1	-1	1	1	-1
χ_7	1	$-i$	i	-1	-1	i	$-i$	1