A-Level H3 Mathematics Solutions (2015-2025)

Thang Pang Ern

This document was last updated on February 26, 2026. If you would like to point out a typo, please send me an email at thangpangern@u.nus.edu. Also, thanks to Kho Thong Liang, Chua Boon Husan, Joel Chang, Teoh Tze Tzun, and Hu Man Keat for contributing.

Contents

Chapter 1

2015 Paper Solutions

Question 1

(i) Since the terms are positive, we take the positive square root to obtain

$$t_{n+2}\sqrt{t_{n+1}} = \sqrt{t_n t_{n+1}}\sqrt{t_{n+1}} = t_{n+1}\sqrt{t_n}.$$

(ii) Define $u_n = t_{n+1}\sqrt{t_n}$. Then, by **(i)**, we have $u_n = u_{n+1}$. Since t_n converges to 1, then

$$\lim_{n\to\infty} t_n = 1 \quad \text{so} \quad \lim_{n\to\infty} u_n = 1.$$

Since the solution to the recurrence relation is $u_n = c$ for some constant c, then $u_n = 1$ for all $n \geq 1$. Hence, $t_{n+1}\sqrt{t_n} = 1$, or equivalently, $t_{n+1} = \frac{1}{\sqrt{t_n}}$.

(iii) From **(ii)**, since $t_{n+1} = t_n^{-1/2}$, then by repeatedly applying this recurrence relation,

$$t_n = t_{n-1}^{-1/2} = \left(t_{n-2}^{-1/2}\right)^{-1/2} = t_{n-2}^{1/4} = t_1^{(-1)^{n-1}/2^{n-1}}.$$

As such,

$$\frac{\ln t_1}{\ln t_1} + \frac{\ln t_2}{\ln t_1} + \frac{\ln t_3}{\ln t_1} + \ldots = \frac{1}{\ln t_1}\sum_{n=1}^{\infty} \ln t_n = \sum_{n=1}^{\infty} \frac{\ln\left(t_1^{(-1)^{n-1}/2^{n-1}}\right)}{\ln(t_1)}$$

By a property of logarithms, we obtain the following geometric series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1} = \frac{2}{3}$$

Question 2

(i) Note that $G = (a_1 \ldots a_n)^{1/n}$. If the two numbers, say $a_i = a$ and $a_j = b$ are replaced by G and $\frac{ab}{G}$ respectively, the new geometric mean is

$$G' = \left[\left(\prod_{k\neq i,j} a_k\right) G \cdot \frac{ab}{G}\right]^{1/n} = \left[\left(\prod_{k\neq i,j} a_k\right) ab\right]^{1/n} = (a_1 \ldots a_n)^{1/n} = G.$$

To show that the sum is not increased, it suffices to show that

$$\frac{ab}{G} + G \leq a + b.$$

This is equivalent to

$$G^2 - (a+b)G + ab \leq 0 \quad \text{or rather} \quad (G-a)(G-b) \leq 0.$$

This holds because $a \geq G \geq b$.

(ii) Let $P(n)$ denote the proposition that $s \geq nG$ for all positive integers n. When $n = 1$, we have the inequality

$$a_1 + a_2 + \ldots + a_n \geq (a_1 \ldots a_n)^{1/n}$$

which holds by the AM-GM inequality.

Suppose $s_n \geq nG_n$ for some positive integer n. Now, consider the case where there are $n+1$ numbers. If all the numbers are equal, we are done. Otherwise, we can find two numbers, say $a_i = a$ and $a_j = b$ such that $a \geq G_{n+1} \geq b$. Replacing them by G_{n+1} and $\frac{ab}{G_{n+1}}$ respectively yields

$$\begin{aligned}
s_{n+1} &= a_1 + a_2 + \ldots + a_n + a_{n+1} \\
&\geq G_{n+1} + \frac{ab}{G_{n+1}} + \sum_{k \neq i,j} a_k \quad \text{by } \textbf{(i)} \\
&= \left(\frac{ab}{G_{n+1}} + \sum_{k \neq i,j} a_k \right) + G_{n+1} \\
&\geq n \left(\frac{a_1 \ldots a_{n+1}}{G_{n+1}} \right)^{1/n} + G_{n+1} \quad \text{by the induction hypothesis} \\
&= n \left(\frac{G_{n+1}^{n+1}}{G_{n+1}} \right)^{1/n} + G_{n+1} \\
&= (n+1) G_{n+1}
\end{aligned}$$

By strong induction, $P(n)$ holds for all positive integers n.

(iii) Applying the AM-GM inequality, we have

$$(1+a_1)(1+a_2)\ldots(1+a_n) \leq \left(\frac{1+a_1+1+a_2+\ldots+1+a_n}{n} \right)^n$$

which is equal to

$$\left(1 + \frac{s}{n}\right)^n \tag{1.1}$$

By the binomial theorem, (1.1) is equal to

$$1 + \binom{n}{1}\frac{s}{n} + \binom{n}{2}\left(\frac{s}{n}\right)^2 + \ldots + \binom{n}{n}\left(\frac{s}{n}\right)^n.$$

For any k, we have

$$\begin{aligned}\binom{n}{k}\left(\frac{s}{n}\right)^k &= \frac{n(n-1)\dots(n-k+1)}{k!}\frac{s^k}{n^k} \\ &= \frac{n(n-1)\dots(n-k+1)}{n^k}\frac{s^k}{k!} \\ &= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\dots\left(\frac{n-k+1}{n}\right)\frac{s^k}{k!} \\ &= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)\frac{s^k}{k!} \\ &\le \frac{s^k}{k!}\end{aligned}$$

where equality holds if and only if $k = 1$.

Question 3

(i) We have

$$\begin{aligned}\int \tan^{n+2} x\, dx + \int \tan^n x\, dx &= \int \tan^n x\left(\tan^2 x + 1\right)\, dx \\ &= \int \tan^n x \sec^2 x\, dx \\ &= \frac{\mathrm{tan^{n+1}x}}{n+1} + c\end{aligned}$$

(ii) We have

$$\begin{aligned}\int \tan^5 x\, dx &= \frac{\tan^4 x}{4} - \int \tan^3 x\, dx \quad \text{by } \textbf{(i)} \\ &= \frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \int \tan x\, dx\right) \\ &= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + c\end{aligned}$$

(iii) **(a)** Using integration by parts, we have

$$\begin{aligned}\int g(x)\, dx &= xg(x) - \int xg'(x)\, dx \\ &= xg(x) - \int f'(x)\, dx \\ &= xg(x) - f(x) + c\end{aligned}$$

(b) Again, using integration by parts, we have

$$\begin{aligned}\int f(x)g(x)\, dx &= xf(x)g(x) - \int xf'(x)g(x)\, dx - \int xf(x)g'(x)\, dx \\ &= xf(x)g(x) - \int x^2 g'(x)g(x) - \int f(x)f'(x)\, dx\end{aligned}$$

This is equal to

$$
\begin{aligned}
&= xf(x)g(x) - \int \left(1+x^2\right) g'(x) g(x)\ dx + \int g(x) g'(x)\ dx - \int f(x) f'(x)\ dx \\
&= xf(x)g(x) - \int h'(x)\ dx + \frac{(g(x))^2}{2} - \frac{(f(x))^2}{2} \\
&= xf(x)g(x) - h(x) + \frac{(g(x))^2}{2} - \frac{(f(x))^2}{2} + c
\end{aligned}
$$

(iv) Consider setting

$$2f(x) = \ln\left(1+x^2\right) \quad \text{and} \quad g(x) = \tan^{-1} x.$$

So,

$$f'(x) = \frac{x}{1+x^2} = xg'(x)$$

and

$$h'(x) = \left(1+x^2\right) g'(x) g(x) = \tan^{-1} x.$$

Hence,

$$h(x) = x\tan^{-1} x - \int \frac{x}{1+x^2}\ dx = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + c.$$

To conclude,

$$
\begin{aligned}
\int \ln\left(1+x^2\right) \tan^{-1} x\ dx &= 2\int f(x) g(x)\ dx \\
&= 2xf(x)g(x) - 2h(x) + (g(x))^2 - (f(x))^2 + c
\end{aligned}
$$

so the answer is

$$x\ln\left(1+x^2\right)\tan^{-1} x - 2x\tan^{-1} x + \ln\left(1+x^2\right) + \left(\tan^{-1} x\right)^2 - \frac{\left(\ln\left(1+x^2\right)\right)^2}{4} + c.$$

Question 4

(i) **(a)** 6^{10}

(b) Let A_i denote the event that a cake of type i is not bought, where $1 \leq i \leq 6$ is a positive integer. From **(a)** the number of ways if there is no restriction is 6^{10}.

If a cake of type i is not bought, there are five remaining types, so $|A_i| = 5^{10}$.

Similarly,

$$
\begin{aligned}
\left|A_i \cap A_j\right| &= 4^{10} \\
\left|A_i \cap A_j \cap A_k\right| &= 3^{10} \\
\left|A_1 \cap A_2 \cap \ldots \cap A_6\right| &= 0
\end{aligned}
$$

We wish to find

$$\left|A_1' \cap A_2' \cap \ldots \cap A_6'\right| = 6^{10} - |A_1 \cup A_2 \cup \ldots \cup A_6|.$$

By the principle of inclusion and exclusion, $|A_1 \cup A_2 \cup \ldots \cup A_6|$ is equal to

$$\begin{aligned}
&= \sum_{i=1}^{6} |A_i| - \sum_{i<j} \left|A_i \cap A_j\right| + \sum_{i<j<k} \left|A_i \cap A_j \cap A_k\right| + \ldots + |A_1 \cap A_2 \cap \ldots \cap A_6| \\
&= \binom{6}{1}5^{10} - \binom{6}{2}4^{10} + \binom{6}{3}3^{10} - \binom{6}{4}2^{10} + \binom{6}{5}1^{10}
\end{aligned}$$

Hence, the required answer is

$$6^{10} - \binom{6}{1}5^{10} + \binom{6}{2}4^{10} - \binom{6}{3}3^{10} + \binom{6}{4}2^{10} - \binom{6}{5}1^{10}$$

(c) There are $\binom{6}{4} = 15$ ways to choose 4 distinct types from six. Then, the number of ways for each type of cake to be bought by at least one customer is

$$4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}2^{10} - \binom{4}{3}1^{10}.$$

By the multiplication principle, the required expression is

$$15\left(4^{10} - \binom{4}{1}3^{10} + \binom{4}{2}2^{10} - \binom{4}{3}1^{10}\right).$$

(ii) **(a)** There are 10 choices for six distinct types of cakes. As such, the problem is analogous to distributing 10 identical objects into 6 distinct boxes. So, the number of distinct selections is

$$\binom{10+6-1}{6-1} = \binom{15}{5} = 3003.$$

(b) If each type of cake is to be included, we put one object into each distinct box first as what we did in **(a)**. Then, we distribute the rest. Hence, the number of distinct selections is

$$\binom{10-6+6-1}{6-1} = \binom{9}{5} = 126.$$

Question 5

(i) **(a)** $x_4 = 5$ and $x_5 = 8$

(b) Consider $n+2$, which is a positive integer. There are two mutually exclusive cases where the composition starts with either 1 or 2.

- **Case 1:** For the case where the composition starts with 1, there are x_{n+1} ways for the remaining $n+1$ to be expressed as a sum of 1s and 2s.
- **Case 2:** For the case where the composition starts with 2, there are x_n ways for the remaining n to be expressed as a sum of 1s and 2s. By the addition principle, it follows that $x_{n+2} = x_{n+1} + x_n$.

(ii) We first construct the recurrence relation for y_n. Similar to **(ib)**, we have $y_{n+3} = y_{n+2} + y_{n+1} + y_n$, where $y_1 = 1$, $y_2 = 2$ and $y_3 = 4$. To show that exactly one of y_{n+2}, y_n is even, it is equivalent to showing that $y_{n+2} + y_n$ is odd. We shall prove this statement by induction.

When $n = 1$, $y_3 + y_1 = 5$ is odd, so the base case is true. Now, suppose $y_{k+2} + y_k$ is odd for some positive integer k. We consider

$$y_{k+3} + y_{k+1} = (y_{k+2} + y_{k+1} + y_k) + y_{k+1} = y_{k+2} + 2y_{k+1} + y_k$$

which is odd as well. By strong induction, it follows that $y_{n+2} + y_n$ is odd for all positive integers n.

Question 6

(i) The differential equation can be written as

$$\frac{dy}{dx} + \frac{y}{x} = 1.$$

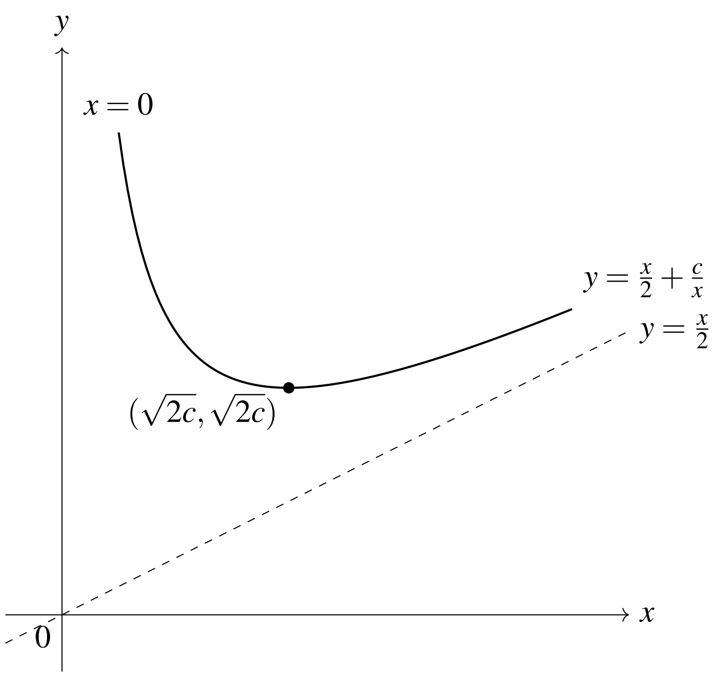
So, the integrating factor is $e^{\int \frac{1}{x}\,dx} = x$. Multiplying both sides of the differential equation by the integrating factor yields

$$\frac{d}{dx}(xy) = x,$$

so $xy = \frac{x^2}{2} + c$. As such, $y = \frac{x}{2} + \frac{c}{x}$. Setting $\frac{dy}{dx} = 0$ yields $\frac{1}{2} - \frac{c}{x^2} = 0$, so $x = \pm\sqrt{2c}$. For the solution curve S to have a minimum point, we must have $c > 0$.

Note that S has a vertical asymptote $x = 0$ and an oblique asymptote $y = \frac{x}{2}$. Also, the minimum point of S lies on the line $x = \sqrt{2c}$. When $x = \sqrt{2c}$, we have $y = \sqrt{2c}$, so the line passing through the origin and $\left(\sqrt{2c}, \sqrt{2c}\right)$ is $y = x$.

(ii)

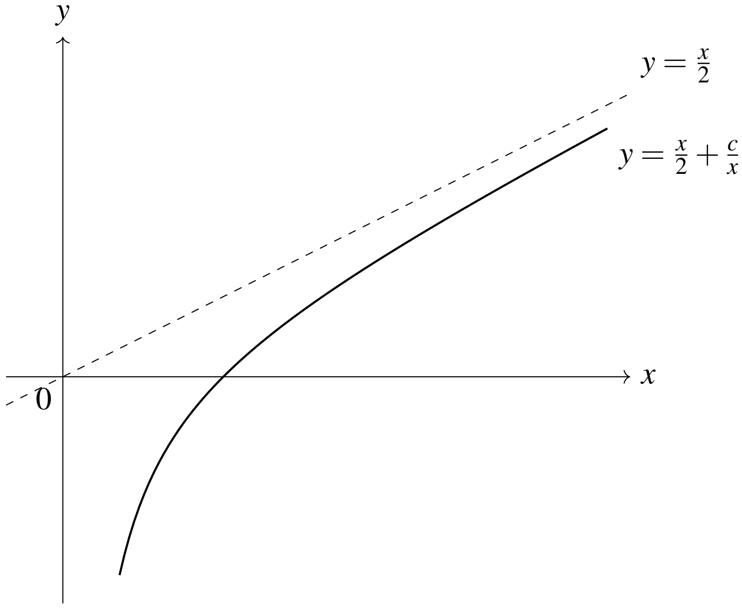


(iii) After enlargement, let the new variables be $X = kx$ and $Y = ky$. Differentiating Y with respect to X yields

$$\frac{dY}{dX} = \frac{dY}{dy}\frac{dy}{dx}\frac{dx}{dX} = (k)\left(1 - \frac{y}{x}\right)\left(\frac{1}{k}\right) = 1 - \frac{y}{x} = \frac{dy}{dx}.$$

So, the enlarged curve is also a solution to the differential equation.

(iv) Recall we obtained the solution $y = \frac{x}{2} + \frac{c}{x}$, where $c > 0$. To obtain a solution curve which is not an enlargement of S, we consider the case where $c < 0$.



Question 7

(i) Dividing both sides by $x + 1$ yields

$$\frac{dy}{dx} + y\left(\frac{2}{x+1}\right) = \frac{e^x}{x+1}.$$

Then, the integrating factor is

$$e^{\int \frac{2}{x+1}\,dx} = (x+1)^2.$$

Multiplying both sides of the differential equation by the integrating factor yields

$$(x+1)^2\frac{dy}{dx} + 2y(x+1) = e^x(x+1).$$

By the product rule,

$$\frac{d}{dx}\left(y(x+1)^2\right) = e^x(x+1).$$

Integrating both sides,

$$y(x+1)^2 = \int e^x(x+1)\,dx = \int xe^x + e^x\,dx = xe^x - \int e^x\,dx + e^x = xe^x + c.$$

Here, c is a constant. So, the general solution is

$$y = \frac{xe^x + c}{(x+1)^2}.$$

(ii) We have

$$\frac{d}{dx}\left(f(x)\frac{dy}{dx}+g(x)y\right)=a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y$$
$$f'(x)\frac{dy}{dx}+f(x)\frac{d^2y}{dx^2}+g'(x)y+g(x)\frac{dy}{dx}=a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y$$

Hence,

$$(f(x)-a(x))\frac{d^2y}{dx^2}+\left(f'(x)+g(x)-b(x)\right)\frac{dy}{dx}+\left(g'(x)-c(x)\right)y=0.$$

As such,

$$a(x)=f(x)$$
$$b(x)=f'(x)+g(x)$$
$$c(x)=g'(x)$$

One checks that $a''(x)-b'(x)+c(x)=0$, so $f(x)=a(x)$, and

$$g(x)=b(x)-f'(x)=b(x)-a'(x).$$

(iii) From the given substitution, differentiating both sides yields

$$\frac{dz}{dx}=f'(x)\frac{dy}{dx}+f(x)\frac{d^2y}{dx^2}+g(x)\frac{dy}{dx}+g'(x)y.$$

By considering **(ii)**,

$$\frac{dz}{dx}=a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y.$$

So, set $a(x)=x^2+x$, $b(x)=4x+1$ and $c(x)=2$. One verifies that $a''(x)-b'(x)+c(x)=0$. As such,

$$\frac{dz}{dx}=(x+1)e^x$$

so $z=xe^x$. The differential equation becomes

$$\left(x^2+x\right)\frac{dy}{dx}+2xy=xe^x.$$

Dividing both sides by x^2+x yields

$$\frac{dy}{dx}+\frac{2}{x+1}y=\frac{e^x}{x+1}.$$

By **(i)**, we obtain the general solution, which is

$$y=\frac{xe^x+c}{(x+1)^2}$$

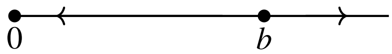
where c is a constant.

Question 8

(i) We first find the equilibrium points. Setting the RHS to be 0, we obtain $aN(N-b) = 0$, which gives equilibrium points at $N = 0$ and $N = b$.

- If $0 < N < b$, then $N - b < 0$, so $aN(N-b) < 0$. As such, N decreases and tends to 0 in the long run.
- If $N > b$, then N and $N - b > 0$, so $an(N-b) > 0$. As such, N increases indefinitely in the long run.

The following is the phase line diagram for the differential equation.



(ii) We have

$$\frac{dN}{dt} = -abN\left(1 - \frac{N}{b}\right).$$

So,

$$\int \frac{1}{N\left(1-\frac{N}{b}\right)}\, dN = -abt + c.$$

Using partial fraction decomposition,

$$\int \frac{1}{N}\, dN + \frac{1}{b}\int \frac{1}{1-\frac{N}{b}}\, dN = -abt + c.$$

Integrating the LHS yields

$$\ln\left|\frac{N}{b-N}\right| = -abt + c$$
$$\frac{N}{b-N} = e^{-abt+c}$$

So,

$$N = \frac{be^{-abt+c}}{1+e^{-abt+c}} = \frac{b}{1+e^{abt-c}}.$$

As t tends to infinity, we must have N tends to 400, so $b = 400$. When $t = 0$, we have $N = 40$, so $e^{-c} = 9$. This implies $c = -\ln 9$. Lastly, when $N = 200$, we have $t = 10\ln 3$ so

$$200 = \frac{400}{1+e^{4000a\ln 3+\ln 9}}.$$

This implies $e^{4000a\ln 3+\ln 9} = 1$, so $a = -\frac{1}{2000}$. We conclude that the expression for N in terms of t is

$$N = \frac{400}{1+9e^{-0.2t}}.$$

Chapter 2

2016 Paper Solutions

Question 1

Suppose we are given the substitution $u = x + \frac{1}{x} = \frac{x^2+1}{x}$. Differentiating both sides with respect to x yields

$$\frac{du}{dx} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}.$$

Also, $u^2 = x^2 + 2 + \frac{1}{x^2}$. Dividing the numerator and the denominator of the integrand by x, we see that the integrand can also be written as

$$\frac{\left(x^2-1\right)/x}{\left(x^2+1\right)\sqrt{x^2+1+1/x^2}}.$$

Since $dx = \frac{x^2}{x^2-1}\,du$, we have

$$\begin{aligned}
\frac{x^2-1}{\left(x^2+1\right)\sqrt{x^4+x^2+1}}\,dx &= \frac{\left(x^2-1\right)/x}{\left(x^2+1\right)\sqrt{x^2+1+1/x^2}} \cdot \frac{x^2}{x^2-1}\,du \\
&= \frac{x}{\left(x^2+1\right)\sqrt{u^2-1}}\,du \\
&= \frac{1}{u\sqrt{u^2-1}}\,du
\end{aligned}$$

We then change the limits of integration. When $x = 1$, we have $u = 2$; when $x = \frac{1}{2}\left(3+\sqrt{5}\right)$, we have $u = 3$. As such, the original integral becomes

$$\int_2^3 \frac{1}{u\sqrt{u^2-1}}\,du.$$

Recall one of the Pythagorean identities $\tan^2\theta + 1 = \sec^2\theta$, so $\tan^2\theta = \sec^2\theta - 1$. As such, we shall use the substitution $u = \sec\theta$, which yields $du = \sec\theta\tan\theta\,d\theta$. The integral becomes

$$\int_{\sec^{-1}(2)}^{\sec^{-1}(3)} \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta-1}}\,d\theta = \int_{\sec^{-1}(2)}^{\sec^{-1}(3)} 1\,d\theta = \sec^{-1}(3) - \sec^{-1}(2).$$

In fact, $\sec^{-1}(2)$ can be expressed as $\frac{\pi}{3}$. To see why, let $y = \sec^{-1}(2)$. Then, $\sec y = 2$, so $\cos y = \frac{1}{2}$. Hence, $y = \frac{\pi}{3}$. It follows that the value of the original integral is $\sec^{-1}(3) - \frac{\pi}{3}$.

Question 2

(i) If f is a linear function, we can write $f(x) = mx + c$. Substituting this into the functional equation yields

$$
\begin{aligned}
m(x+2) + c &= a[m(x+1)+c] - (mx+c) \\
mx + 2m + c &= amx + am + ac - mx - c \\
(2m - am)x + 2m + 2c - am - ac &= 0
\end{aligned}
$$

Since this equation must hold for all values of x, then

$$
\begin{aligned}
2m - am &= 0 \quad 1 \\
2m + 2c - am - ac &= 0 \quad 2
\end{aligned}
$$

From 1, either $m = 0$ or $a = 2$. If $m = 0$, substituting this into 2 yields $2c - ac = 0$, so either $c = 0$ or $a = 2$. On the other hand, if $a = 2$, substituting this into 2 yields $2m + 2c - 2m - 2c = 0$, which indeed holds for all m and c.

As such, $(m, c, a) = (0, 0, a)$ or $(m, c, 2)$. In other words, either $f(x) = 0$ which implies $a \in \mathbb{R}$ is arbitrary, or $f(x) = mx + c$, where $m, c \in \mathbb{R}$ are arbitrary, which implies $a = 2$.

(ii) Given that $f(0) = 0$ and $|f(1)| = 1$, let $P(n)$ be the proposition that

$$f(n-1)f(n+1) + 1 = [f(n)]^2 \text{ for all positive integers } n.$$

When $n = 1$, the LHS evaluates to $f(0)f(2) + 1 = 1$, whereas the RHS evaluates to $[f(1)]^2 = 1$, so the base case P_1 is true.

Assume that $P(k)$ is true for some positive integer k. That is,

$$f(k-1)f(k+1) + 1 = [f(k)]^2.$$

We wish to prove that $P(k+1)$ is true. That is,

$$f(k)f(k+2) + 1 = [f(k+1)]^2$$

By replacing x with n in the original functional equation, we have $f(n+2) = af(n+1) - f(n)$. So,

$$
\begin{aligned}
f(k)f(k+2) + 1 &= f(k)[af(k+1) - f(k)] + 1 \\
&= af(k)f(k+1) - [f(k)]^2 + 1 \\
&= af(k)f(k+1) - f(k-1)f(k+1) \\
&= f(k+1)[af(k) - f(k-1)] \\
&= [f(k+1)]^2
\end{aligned}
$$

Since $P(1)$ is true and $P(k)$ is true implies $P(k+1)$ is true, then $P(n)$ is true for all positive integers n.

(iii) Suppose $a = 2$. Then,

$$f(x+2) = 2f(x+1) - f(x)$$
$$f(x+2) - f(x+1) = f(x+1) - f(x)$$

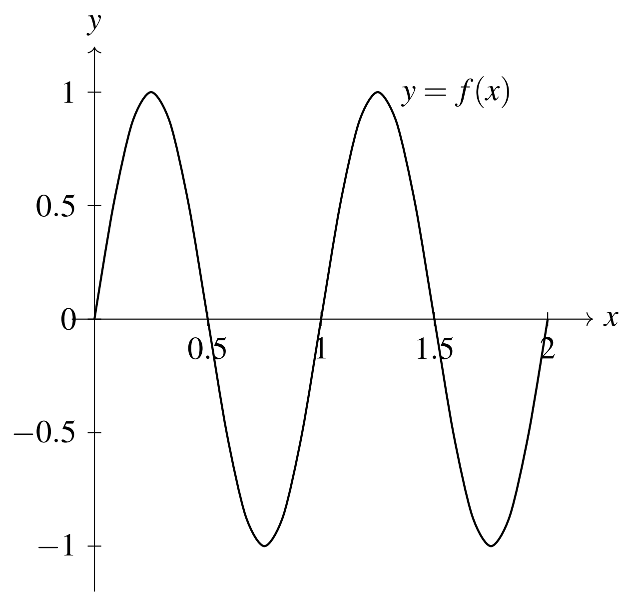
Define $g(x) = f(x+1) - f(x)$, so $g(x+1) = g(x)$, which means that g is a periodic function with period 1. For example, we can choose $g(x) = \sin(2\pi x)$. Note that

$$\sin[2\pi(x+1)] = \sin(2\pi x)\cos 2\pi + \cos(2\pi x)\sin 2\pi = \sin(2\pi x)$$

so

$$\sin[2\pi(x+1)] - \sin(2\pi x) = 0.$$

This suggests that we can also set $f(x) = \sin(2\pi x)$, which is a non-linear function. Here is a sketch of f.



Question 3

(i) Since x_n and y_n are increasing sequences of numbers, regardless of whether $i \leq j$ or $i \geq j$, $\left(x_i - x_j\right)\left(y_i - y_j\right)$ is either the product of two non-negative numbers or two non-positive numbers. So, we must have $(x_i - x_j)(y_i - y_j) \geq 0$. As such,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)(y_i - y_j) \geq 0$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_iy_i - x_iy_j - x_jy_i + x_jy_j) \geq 0$$
$$n\sum_{i=1}^{n}x_iy_i - \left(\sum_{i=1}^{n}x_i\right)\left(\sum_{j=1}^{n}y_j\right) - \left(\sum_{j=1}^{n}x_j\right)\left(\sum_{i=1}^{n}y_i\right) + n\sum_{j=1}^{n}x_jy_j \geq 0$$
$$2n\sum_{i=1}^{n}x_iy_i \geq 2\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)$$
$$\sum_{i=1}^{n}x_iy_i \geq \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)\left(\sum_{i=1}^{n}y_i\right)$$

(ii) Based on the inequality that we need to deduce, it suggests that we need to set $n = 3$ and use the fact that $A + B + C = \pi$. Let $x_1 = a, x_2 = b, x_3 = c$ and $y_1 = A, y_2 = B, y_3 = C$. Based on the diagram, x_n and y_n are increasing sequences. Applying the inequality in **(i)** yields

$$aA + bB + cC \geq \frac{1}{3}(a+b+c)(A+B+C)$$
$$aA + bB + cC \geq \frac{1}{3}\pi(a+b+c)$$

(iii) Suppose

$$\{x_i\} = \left\{\frac{a+b}{c}, \frac{c+a}{b}, \frac{b+c}{a}\right\}.$$

It suggests that

$$\{y_i\} = \left\{a^2+b^2, c^2+a^2, b^2+c^2\right\}.$$

By using **(i)**, we have

$$\begin{aligned}
\sum_{i=1}^{3}x_iy_i &= \frac{(a+b)(a^2+b^2)}{c} + \frac{(c+a)(c^2+a^2)}{b} + \frac{(b+c)(b^2+c^2)}{a} \\
&\geq \frac{1}{3}\left(\frac{a+b}{c} + \frac{c+a}{b} + \frac{b+c}{a}\right)(a^2+b^2+c^2+a^2+b^2+c^2) \\
&= \frac{2}{3}\left(\frac{a+b}{c} + \frac{c+a}{b} + \frac{b+c}{a}\right) \text{ since } a^2+b^2+c^2 = 1 \\
&= \frac{2}{3}\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}\right) \\
&\geq \frac{2}{3}(2+2+2) \text{ by the AM-GM inequality} \\
&= 4
\end{aligned}$$

Here, equality is attained if and only if $\frac{a}{b} = \frac{b}{a}$, $\frac{a}{c} = \frac{c}{a}$ and $\frac{b}{c} = \frac{c}{b}$. Equivalently, $a^2 = b^2 = c^2$. Since $a^2 + b^2 + c^2 = 1$, we must have $a, b, c = \pm\frac{1}{\sqrt{3}}$.

Remark for Question 3: The inequality mentioned in **(i)** is known as Chebyshev's sum inequality.

Question 4

(i) There are 9 choices for the digits in the hundreds place $(1,\ldots,9)$, 9 choices for the digits in the tens place $(1,\ldots,9$ but excluding the choice in the hundreds place) and 8 choices for the digits in the ones place $(1,\ldots,9$ but excluding the choice in the hundreds and tens place). So, there are $9\times 8\times 7 = 504$ numbers in S.

(ii) Each digit $1,\ldots,9$ appears in the hundreds place $\frac{504}{9} = 56$ times. Similarly, each digit appears 56 times in the tens place and 56 times in the ones place. As such, the contribution per place value is

$$56\,(1+2+\ldots+9) = 56\times 45 = 2520.$$

The total sum required is

$$2520\times 100 + 2520\times 10 + 2520\times 1 = 279720.$$

(iii) **(a)** The digits in the ones place must be even, so there are 4 choices. As such, there are 8 choices for the digits in the tens place, and consequently 7 choices for the digits in the hundreds place. So, there are $4\times 8\times 7 = 224$ numbers divisible by S.

(b) Recall that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. We consider two cases.

- **Case 1:** Suppose all digits are from the same congruence class.

  The digits which are congruent to 0 mod 3 are $3,6,9$. Since we have to choose all three digits, there is only 1 combination. There are $3! = 6$ permutations. Next, the digits which are congruent to 1 mod 3 are $1,4,7$. Similarly, there is only 1 combination and $3! = 6$ permutations. We have a similar argument for the numbers which are congruent to 2 mod 3 (the numbers are $2,5,8$). Again, there is only 1 combination and $3! = 6$ permutations.

  The total permutations in this case is $6+6+6 = 18$.
- **Case 2:** Suppose we have one digit from each congruence class. There are $3^3 = 27$ combinations and there are $3! = 6$ permutations for each combination. This yields a total of $27\times 6 = 162$ permutations.

To conclude, there are $18+162 = 180$ numbers in S which are divisible by 3.

(c) Let A,B,C denote the following sets:

$$A = \{n\in S : n \text{ divisible by } 2\}$$
$$B = \{n\in S : n \text{ divisible by } 3\}$$
$$C = \{n\in S : n \text{ divisible by } 5\}$$

We wish to find $|A' \cap B' \cap B'|$.

We have

$$\left|A' \cap B' \cap C'\right| = |S| - |A \cup B \cup C|$$

where $|S| = 504$ by **(i)**. By the principle of inclusion and exclusion,

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 224 + 180 + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \text{ by } \textbf{(a)} \text{ and } \textbf{(b)}\end{aligned}$$

We then compute $|C|$. Note that a number is divisible by 5 if and only if its ones digit is 5. As such, there is 1 choice for the digit in the ones place, 8 choices for the digit in the tens place and 7 choices for the digit in the hundreds place. This implies $|C| = 56$.

$A \cap B$ is the set of numbers in S which are divisible by both 2 and 3, and hence divisible by 6. So, the ones place can have the even digits $2, 4, 6, 8$ and the sum of digits must be divisible by 3.

- **Case 1:** Suppose the ones digit is 2. Let H and T be the digits in the hundreds and tens place respectively. Then, $H + T \equiv 1 \pmod 3$. We can have either
  - **Subcase 1:** If $H \equiv 1 \pmod 3$ and $T \equiv 0 \pmod 3$, then the choices for H are $3, 6, 9$; the choices of T are $1, 4, 7$. This yields $3 \times 3 = 9$ pairs.
  - **Subcase 2:** If $H \equiv 0 \pmod 3$ and $T \equiv 1 \pmod 3$, similar to subcase 1, this yields 9 pairs.
  - **Subcase 3:** If $H \equiv 2 \pmod 3$ and $T \equiv 2 \pmod 3$, the choices for H are $5, 8$; the choices for T are $5, 8$. However, we need to exclude pairs where $H = T$, so the possible pairs are $(5, 8), (8, 5)$. The total number of pairs is 2.

  So, case 1 has 20 valid numbers.
- **Case 2:** Ones digit is 4
- **Case 3:** Ones digit is 6
- **Case 4:** Ones digit is 8

The other three cases are very similar to case 1, and each case consists of 20 valid numbers. As such, $|A \cap B| = 80$.

$A \cap C$ consists the numbers which are divisible by both 2 and 5, and numbers divisible by 2 and 5 are also divisible by 10. Such numbers would end with 0, which is not possible as the digits in S are non-zero. So, $|A \cap C| = 0$. Consequently, $|A \cap B \cap C| = 0$.

$B \cap C$ consists of the numbers that are divisible by both 3 and 5, and numbers divisible by both 3 and 5 are divisible by 15. There are not many numbers to consider, so through systematic listing, one would be able to deduce that $|B \cap C| = 20$.

Putting everything together,

$$|A \cup B \cup C| = 224 + 180 + 56 - 80 - 20 - 0 + 0 = 360$$

so $|A' \cap B' \cap C'| = 504 - 360 = 144$.

Question 5

(i) **(a)** The only possible cases are that one of the boxes contains exactly k objects, where $k = 1, \ldots, \frac{1}{2}r$.

(b) $P(r,2) = \frac{1}{2}(r-1)$

(ii) We shall consider two cases.

- **Case 1:** Suppose one of the boxes contains exactly 1 object. Then, the number of ways is $P(r-1, n-1)$.
- **Case 2:** Suppose all boxes contain at least 2 objects. Then, we place 1 object into each of the n boxes. We then distribute the remaining $r-n$ objects into the n boxes such that each box will have at least one of these objects. Thus, each box will have at least 2 objects in it. The number of ways is $P(r-n, n)$.

Since the cases are mutually exclusive, then $P(r,n) = P(r-1, n-1) + P(r-n, n)$ for all $1 < n \leq r$.

(i) For sufficiently large m, by repeatedly applying the recurrence relation in **(ii)**, we obtain

$$\begin{aligned} P(m+k, m) &= P(m-1+k, m-1) + P(k, m) \\ &= P(m-2+k, m-2) + P(k, m-1) + P(k, m) \end{aligned}$$

and eventually, we have

$$P(m+k, m) = P(k,1) + P(k,2) + \ldots + P(k,m) = \sum_{r=1}^{m} P(k,r)$$

Since k is fixed, we must have

$$\sum_{r=1}^{m} P(k,r) = \sum_{r=1}^{k} P(k,r)$$

as $P(k,r) = 0$ for all $r > k$. We conclude that $P(m+k, m)$ eventually reaches a constant value.

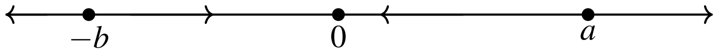
(ii) For

$$P(m+k, m) = P(m-1+k, m-1),$$

we need $P(k,m) = 0$, so the least value of m is $k+1$.

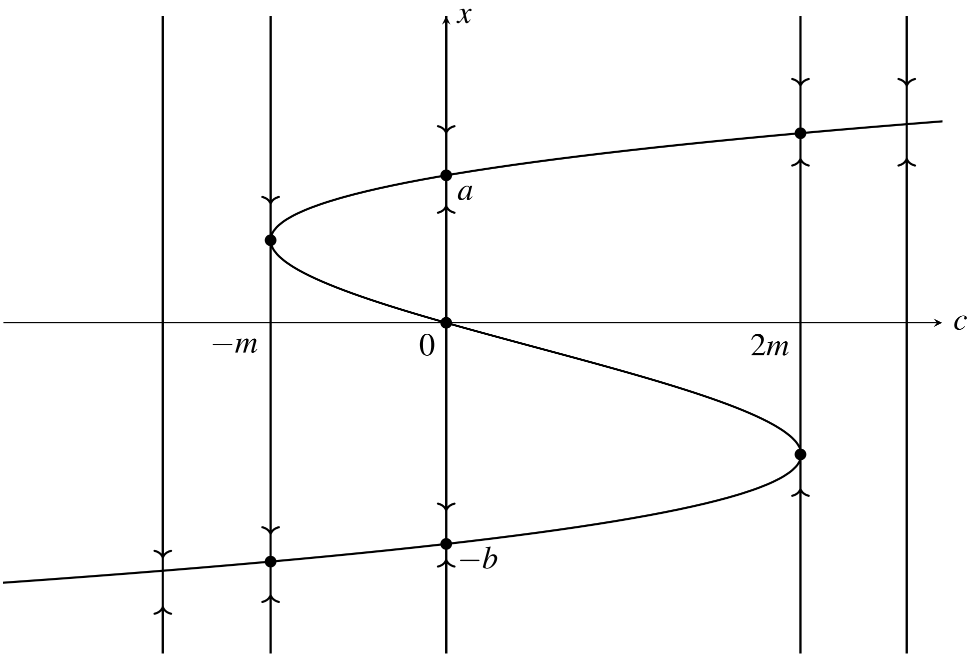
Question 6

(i) The phase line diagram is as follows:



$x = b$ and $x = a$ are unstable; $x = 0$ is stable.

(ii) Setting $\frac{dx}{dt} = 0$ yields $f(x) = -c$. From the graph, $-c = m$ and $-c = -2m$ are critical values, so $c = -m$ and $c = 2m$ are bifurcation values.



(iii) From

$$\frac{dP}{dt} = f(P) \text{ to } \frac{dP}{dt} = f(P) + c,$$

we see that the curve is translated in the positive y-direction by c units. We consider three cases.

- **Case 1:** If $c < -m$, then $f(P) < 0$ for all positive values of P. Consequently, the population decreases over time and eventually reaches 0.
- **Case 2:** If $c > -m$, then there exists some positive root r. If the initial population is non-zero, then P will tend to and stabilise at r in the long run. This happens because $f(P) > 0$ for $P < r$, causing the population to increase, whereas $f(P) < 0$ for $P > r$, causing it to decrease.
- **Case 3:** If $c = -m$, then there exists a positive root r but it is an unstable point. If $0 < P < r$, then the population decreases and eventually goes to 0, otherwise P tends to r.

Question 7

(i) Note that

$$f(x_0, y_0) = 4 - \frac{1}{5+0} = 3.8$$

so we have

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.5(3.8) = 2.9$$

Next,

$$f(x_1, y_1) = 4 - \frac{2.9}{5+0.5} = 3.4727$$

so

$$y_2 = y_1 + hf(x_1, y_1) = 2.9 + 0.5(3.4727) = 4.6364 \approx 4.64.$$

(ii) We have

$$\tilde{y}_{0.5} = 2.8182 + 0.5(3.4876) = 4.5620.$$

So,

$$f(1, \tilde{y}_{0.5}) = 4 - \frac{4.5620}{6} = 3.2397$$

which implies

$$y_1 = 2.8182 + \frac{0.5}{2}(3.4876 + 3.2397) = 4.5000.$$

Consequently,

$$\frac{\Delta y}{\Delta x} = \frac{4.5000 - 2.8182}{1 - 0.5} = 3.3636.$$

which completes the following table:

x	y	$4 - \frac{y}{5+x}$	$\tilde{y}$	$\frac{\Delta y}{\Delta x}$
0	1	3.8	2.9	3.6364
0.5	2.8182	3.4876	4.5620	3.3636
1	4.5000			

(iii) We have

$$\frac{dy}{dx} + \frac{y}{5+x} = 4.$$

The integrating factor is

$$e^{\int \frac{1}{5+x}\,dx} = e^{\ln(5+x)} = 5 + x.$$

Multiplying both sides of the differential equation by the integrating factor yields

$$\frac{d}{dx}\left[y(5+x)\right] = 4(5+x)$$

so

$$y(5+x) = 4\int 5+x\,dx = 4\left(5x+\frac{1}{2}x^2\right)+c.$$

When $x=0$, we have $y=1$, so $c=5$. So,

$$y = \frac{4}{5+x}\left(5x+\frac{1}{2}x^2\right)+\frac{5}{5+x}.$$

When $x=1$, $y=4.5$.

Question 8

(i) **(a)** The auxiliary equation is $m^2+2\gamma\omega m+\omega^2=0$. Treating this as a quadratic in m, its discriminant Δ is

$$4\gamma^2\omega^2-4\omega^2 = 4\omega^2(\gamma+1)(\gamma-1).$$

Note that $\omega,\gamma>0$. If $\Delta<0$, then $0<\gamma<1$, which implies that the system is underdamped. Consequently, it would exhibit oscillatory motion, i.e. system oscillates with a gradually decreasing amplitude.

If $\Delta>0$, then $\gamma>1$. The system does not oscillate and it slowly returns to equilibrium.

If $\Delta=0$, then $\gamma=1$, so the system does not oscillate. It returns to equilibrium as quickly as possible.

(b) When $\omega=5$ and $\gamma=0.8$, the differential equation becomes $\ddot{x}+8\dot{x}+25x=0$. The auxiliary equation is $m^2+8m+25=0$. The roots are $-4\pm 3i$. As such, the solution is of the form

$$x(t) = e^{-4t}(A\cos 3t+B\sin 3t).$$

(ii) Using the substitution $u=x^2y$, differentiating both sides yields

$$\frac{du}{dx} = x^2\frac{dy}{dx}+2xy.$$

Differentiating one more time yields

$$\frac{d^2u}{dx^2} = x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y.$$

The differential equation becomes

$$\begin{aligned}\frac{d^2u}{dx^2}-4x\frac{dy}{dx}-2y+8x^2\frac{dy}{dx}+4x\frac{dy}{dx}+25x^2y+16xy+2y=0\\ \frac{d^2u}{dx^2}+8\frac{du}{dx}+25u=0\end{aligned}$$

The auxiliary equation is $m^2+8m+25=0$. By **(ib)**, the general solution is

$$u(x)=e^{-4x}(A\cos 3x+B\sin 3x).$$

When $x=\frac{\pi}{2}$ and $y=0$, we have $u=0$; when $x=\pi$ and $y=-\frac{1}{\pi^2}$, we have $u=-1$. As such, $u\left(\frac{\pi}{2}\right)=0$ and $u(\pi)=-1$. As such,

$$e^{-2\pi}(-B)=0 \text{ and } e^{-4\pi}(-A)=-1.$$

Consequently, $B=0$ and $A=e^{4\pi}$. So, $u(x)=e^{4\pi-4x}\cos 3x$. We conclude that

$$y(x)=\frac{e^{4\pi-4x}\cos 3x}{x^2}.$$

Chapter 3

2017 Paper Solutions

Question 1

(i) Consider the graph of $y = \ln x$ as shown in Figure 3.1. Plot the points $A\,(a, \ln a)$ and $B\,(b, \ln b)$ and without loss of generality, assume $0 < a \leq b$.

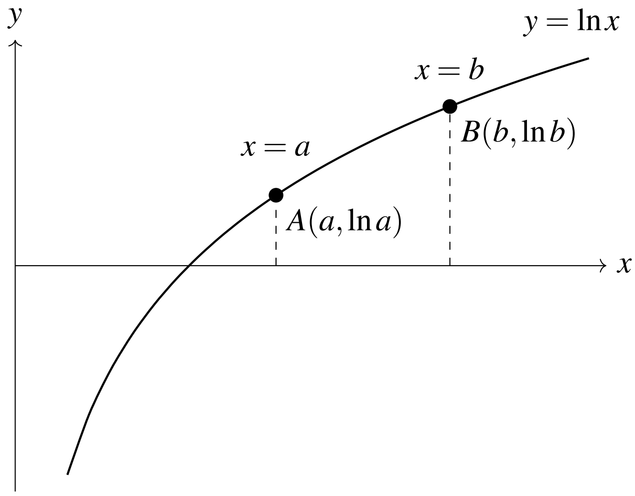


Figure 3.1:

Let M be on the graph such that its x-coordinate is the average of A and B. So, M has coordinates

$$\left(\frac{a+b}{2}, \ln\left(\frac{a+b}{2}\right)\right).$$

Also, let C be such that its y-coordinate is the average of A and B. Then, the y-coordinate of C is

$$\frac{1}{2}\left(\ln a + \ln b\right).$$

As $y = \ln x$ is concave down, the y-coordinate of C is less than or equal to that of M. As such, the result follows with equality attained if and only if $a = b$.

(ii) We have

$$\ln\left(\frac{a+b}{2}\right) \geq \ln\sqrt{ab}.$$

Since $y = \ln x$ is an injective function, then $\frac{a+b}{2} \geq \sqrt{ab}$, which is the AM-GM inequality for two variables.

(iii) Let $y = x\ln x$. Then, $\frac{d^2y}{dx^2} = \frac{1}{x}$ so for all $x > 0$, $\frac{d^2y}{dx^2} > 0$. This shows that y is concave up (alternatively, one can use a graphing calculator to verify this). Let A and B have coordinates $(a, a\ln a)$ and $(b, b\ln b)$ respectively. So,

$$\begin{aligned}
\frac{a\ln a + b\ln b}{2} &\geq \frac{a+b}{2}\ln\left(\frac{a+b}{2}\right) \quad \text{since } y = x\ln x \text{ is concave down}\\
a\ln a + b\ln b &\geq (a+b)\ln\left(\frac{a+b}{2}\right)\\
\ln\left(a^a b^b\right) &\geq \ln\left(\left(\frac{a+b}{2}\right)^{a+b}\right)\\
a^a b^b &\geq \left(\frac{a+b}{2}\right)^{a+b} \qquad \text{by injectivity of } \ln x
\end{aligned}$$

Question 2

(i) Let P_n be the proposition that

$$\frac{d^n}{dx^n}(xy) = x\frac{d^n y}{dx^n} + n\frac{d^{n-1}y}{dx^{n-1}}$$

for all positive integers n.

When $n = 1$, the LHS is $\frac{d}{dx}(xy)$, which is equal to $x\frac{dy}{dx} + y$. This expression is equal to the RHS.

Assume that P_k is true for some positive integer k. That is,

$$\frac{d^k}{dx^k}(xy) = x\frac{d^k y}{dx^k} + k\frac{d^{k-1}y}{dx^{k-1}}.$$

To show that P_{k+1} is true, we need to prove that

$$\frac{d^{k+1}}{dx^{k+1}}(xy) = x\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k}.$$

So,

$$\begin{aligned}
\text{LHS} &= \frac{d^{k+1}}{dx^{k+1}}(xy)\\
&= \frac{d}{dx}\left(\frac{d^k}{dx^k}(xy)\right)\\
&= \frac{d}{dx}\left(x\frac{d^k y}{dx^k} + n\frac{d^{k-1}y}{dx^{k-1}}\right) \quad \text{by induction hypothesis}\\
&= x\frac{d^{k+1}y}{dx^{k+1}} + \frac{d^k y}{dx^k} + n\frac{d^k y}{dx^k}\\
&= x\frac{d^{k+1}y}{dx^{k+1}} + (k+1)\frac{d^k y}{dx^k} = \text{RHS}
\end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, by mathematical induction, P_n is true for all positive integers n.

(ii) **(a)** We have $y_0 = 1$ and

$$y_1 = e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = e^{x^2}\left(-2xe^{-x^2}\right) = -2x$$

and

$$y_2 = e^{x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2}\frac{d}{dx}\left(-2xe^{-x^2}\right) = 4x^2 - 2.$$

(b) We see that $y_{n+2} + 2xy_{n+1} + 2(n+1)y_n$ is equal to

$$e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2xe^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + 2(n+1)e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

or equivalently,

$$e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2e^{x^2}\left[x\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + (n+1)\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right]. \quad (3.1)$$

By considering **(i)** and setting $y = e^{-x^2}$, (3.1) can be simplified as follows:

$$e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(xe^{-x^2}\right) = e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left[\frac{d}{dx}\left(e^{-x^2}\right) + 2xe^{-x^2}\right] = 0$$

(c) From **(b)**, $y_{n+2} + 2xy_{n+1} = -2(n+1)y_n$. Hence,

$$\begin{aligned}\frac{d}{dx}(y_{n+1}) &= \frac{d}{dx}\left[e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\right] \\ &= e^{x^2}\frac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right) + 2xe^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) \\ &= y_{n+2} + 2xy_{n+1} \\ &= -2(n+1)y_n\end{aligned}$$

Question 3

(a) As $\gcd(1591, 3913, 9331) = 43$, factorising 43 from both sides of the equation yields $37x + 91y = 217$, or rather, $37x = 7(31 - 13y)$. So, $37x$ is a multiple of 7, which forces x to be a multiple of 7. The only prime that is a multiple of 7 is 7, but if $x = 7$, then $y = -\frac{6}{13} \notin \mathbb{Z}$. So, we conclude that there are no integer solutions with x prime.

(b) **(i)** As a and b are factors of n, there exist $\lambda, \mu \in \mathbb{Z}$ such that $n = \lambda a = \mu b$. Given that $ra + sb = 1$, then $ran + sbn = n$. So, $ab(r\mu + s\lambda) = n$, which asserts that ab is a factor of n.

(ii) Suppose $x \equiv u \pmod{a}$. Then, there exists $k \in \mathbb{Z}$ such that $x = ka + u$. Write $k = k_1 b + q$ for some $k_1, q \in \mathbb{Z}$. So, $x = ak_1 b + aq + u$, which implies that $x = aq + u \pmod{b}$. As $ra + sb = 1$, then

$$\begin{aligned}r(v-u)a + s(v-u)b &= v - u \\ r(v-u)a &\equiv v - u \pmod{b} \quad (*)\end{aligned}$$

By choosing $k = k_1 b + r(v-u)$, i.e. $q = r(v-u)$, we have

$$\begin{aligned} x &\equiv ar(v-u) + u \pmod{b} \\ &= v - u + u \pmod{b} \quad \text{using } (*) \\ &\equiv v \pmod{b} \end{aligned}$$

Hence, we have constructed a number $x = (b + r(v-u))a + u = ab + ar(v-u) + u$ such that $x \equiv u \pmod{a}$ and $x \equiv v \pmod{b}$.

Question 4

(i)

$$\begin{aligned} I_n + I_{n-2} &= \int_0^{\frac{\pi}{4}} \tan^n x + \tan^{n-2} x \, dx \\ &= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \left(1 + \tan^2 x\right) \, dx \\ &= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx \\ &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n-1} \end{aligned}$$

(ii) $y = \tan x$ is strictly increasing on $[0, \frac{1}{4}\pi]$. Substituting the x-coordinates of the end-points, $0 \le \tan x \le 1$.
Consider the y-coordinates of a linear function $y = mx$ to be upper bounds for all y values of $y = \tan x$. For $0 \le x \le \frac{\pi}{4}$, it must satisfy $0 \le mx \le 1$.

Hence, $0 \le mx \le \frac{m}{4}\pi$, implying that $m = \frac{4}{\pi}$. The required linear function is $y = \frac{4}{\pi}x$. The inequality $\tan x \le \frac{4}{\pi}x$ is true and equality holds if and only if $x = 0$ or $x = \frac{\pi}{4}$.

(iii) Since $\tan x \ge 0$ on $[0, \frac{1}{4}\pi]$, combining this with **(ii)** yields $0 \le \tan x \le \frac{4}{\pi}x$.

So,

$$\begin{aligned} 0 &\le \int_0^{\frac{\pi}{4}} \tan^n x \, dx \le \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi}x\right)^n dx \\ 0 &\le I_n \le \left(\frac{4}{\pi}\right)^n \int_0^{\frac{\pi}{4}} x^n \, dx \\ 0 &\le I_n \le \left(\frac{4}{\pi}\right)^n \left[\frac{x^{n+1}}{n+1}\right]_0^{\frac{\pi}{4}} \\ 0 &\le I_n \le \frac{\pi}{4(n+1)} \end{aligned}$$

As $\lim_{n\to\infty} \frac{\pi}{4(n+1)} = 0$, by the squeeze theorem, I_n tends to zero as well.

(iv) Note that $I_2+I_0=\frac{1}{1}$, $I_4+I_2=\frac{1}{3}$ and $I_6+I_4=\frac{1}{5}$, which are the magnitudes of the first three terms of the series.
By the method of difference,

$$\begin{aligned}\frac{1}{1}-\frac{1}{3}+\frac{1}{5}+\ldots &= (I_2+I_0)-(I_4-I_2)+(I_6-I_4)+\ldots \\ &= I_0=\frac{\pi}{4}\end{aligned}$$

Remark for Question 4: This question involves proving Madhava's formula for π. It is an example of a Madhava series which is a collection of infinite series believed to have been discovered by Madhava of Sangamagrama in the 1200s. James Gregory and Gottfried Wilhelm Leibniz discovered the series much later in the 1670s. For most of the Western world, the series is known as the Leibniz series.

Question 5

(i) Suppose there are no restrictions. For each object, it can go into either box. There are 2^r ways to do this. As there are 2 cases where either box is empty, the result follows.

(ii) **(a)** Note that $S(r,n)$ represents a Stirling number of the second kind.
Let the set M comprise the r objects. So, we write $M=\{a_1,\ldots,a_r\}$.

- **Case 1:** Suppose for some $1\leq j\leq r$, a_j is the only object in a box. There is 1 way as the boxes are identical. The remaining $r-1$ objects can be distributed into the remaining 2 boxes. The number of ways is $2^{r-2}-1$.
- **Case 2:** Suppose for some $1\leq j\leq r$, a_j is mixed with other objects. We first distribute the remaining $r-1$ objects into 3 boxes. Then, a_j can enter either one of the 3 boxes in $3S(r-1,3)$ ways.

Since the two cases are mutually exclusive, the result follows by the addition principle.

(b) As $S(r,3)=2^{r-2}-1+3S(r-1,3)$, then $S(r+2,3)=9S(r,3)+5\left(2^{r-1}\right)-4$.
Let P_r be the proposition that

$$S(r,3)\equiv\begin{cases}0 \pmod 6 & \text{if } r \text{ is even}\\ 1 \pmod 6 & \text{if } r \text{ is odd}\end{cases}$$

for all positive integers r such that $r\geq 3$.

When $r=3$, there is only 1 way to distribute 1 object into 1 box, so $S(3,3)=1\equiv 1 \pmod 6$.
When $r=4$, using the recurrence relation established in **(iia)**, we have $S(4,3)=3+3S(3,3)\equiv 0 \pmod 6$.
These assert that the base cases P_3 and P_4 are true.

Assume P_k is true for some positive integer k such that $k \geq 3$. That is,

$$S(k,3) \equiv \begin{cases} 0 \pmod 6 & \text{if } k \text{ is even} \\ 1 \pmod 6 & \text{if } k \text{ is odd} \end{cases}$$

We wish to prove that P_{k+2} is true. That is,

$$S(k+2,3) \equiv \begin{cases} 0 \pmod 6 & \text{if } k+2 \text{ is even} \\ 1 \pmod 6 & \text{if } k+2 \text{ is odd} \end{cases}$$

Suppose k is even. Then, $k+2$ is also even, so

$$\begin{aligned} S(k+2,3) &= 9S(k,3) + 5\left(2^{k-1}\right) - 4 \\ &\equiv 5\left(2^{k-1}\right) - 4 \pmod 6 \quad \text{by induction hypothesis} \\ &\equiv 0 \pmod 6 \end{aligned}$$

Now, suppose k is odd. Then, $k+2$ is also odd, so

$$\begin{aligned} S(k+2,3) &= 9S(k,3) + 5\left(2^{k-1}\right) - 4 \\ &\equiv 5 + 5\left(2^{k-1}\right) \pmod 6 \quad \text{by induction hypothesis} \\ &= 5\left(2^{k-1}+1\right) \\ &\equiv 5(-1) \pmod 6 \\ &\equiv 1 \pmod 6 \end{aligned}$$

Since P_3 and P_4 are true and P_k is true implies P_{k+1} is true, by mathematical induction, P_r is true for all positive integers r such that $r \geq 3$.

Question 6

(a) **(i)** Label the beads $a_1, \ldots, a_n$.

We first consider a linear permutation, which can be done in $n!$ ways. As the circle can be rotated, then suppose a_1 goes to the old position of a_2, a_2 goes to the old position of a_3, and so on. We obtain a permutation of the same configuration as before. So, the number of arrangements in a circle is $\frac{n!}{n} = (n-1)!$.

(ii) When there are no restrictions, there are $(n-1)!$ ways to arrange the beads. If two beads are adjacent, there are $2((n-1)-1)!$ ways to arrange them. Hence, the required number of ways is $(n-1)! - 2(n-2)! = (n-2)!(n-3)$.

(iii) First, note that the result holds if and only if $n > 5$. There are $\binom{n}{3}$ ways to choose 3 beads out of n and n ways to choose 3 adjacent beads. For

2 fixed but adjacent beads, there are n ways to choose and $n-4$ ways to choose the 3rd bead so that the 3rd bead is not adjacent to the first two beads.

Hence, the answer is

$$\begin{aligned}\binom{n}{3}-n-n(n-4) &= \frac{n(n-1)(n-2)}{6}-n-n(n-4)\\ &= n\left[\frac{(n-1)(n-2)+6(3-n)}{6}\right]\\ &= \frac{n(n-4)(n-5)}{6}\end{aligned}$$

(b) Let A and B denote the following sets:

$A = \{$all 4-tuples denoting all collections of 4 points on the perimeter of the circle$\}$
$B = \{$all interior points in the circle when the maximum possible number of interior points is achieved$\}$

Let $f : A \to B$ be a function. Suppose $a \in A$. Then there exists $b \in B$ such that $f(a) = b$ as we can always find 4 points that form the 2 chords on which b is the intersection of the 2 chords.

To show f is injective, suppose $f(a) = f(a')$. Suppose on the contrary that $a \neq a'$. Then, we can shift 2 of the 4 points in a' such that 2 additional interior points are formed instead of 1, which is a contradiction. So, $a = a'$.

To show f is surjective, as every $b \in B$ is formed by the intersection of 2 chords, it corresponds to 4 distinct points on the perimeter of the circle.

Since f is injective and surjective, it is thus bijective, so by the bijection principle, $|A| = |B| = \binom{n}{4}$.

Question 7

(i) Let $x = 4k+3$ for some $k \in \mathbb{Z}_{\geq 0}$. As $x \equiv 1 \pmod 2$, then x is odd, so its divisors are also odd.
Suppose on the contrary that all the prime divisors of x are of the form 1 mod 4. For any two integers of the form 1 mod 4, say $4m+1$ and $4n+1$, where $m, n \in \mathbb{Z}$, their product, $16mn+4m+4n+1$ is also 1 mod 4. Hence, the product of any number of integers of the form 1 mod 4 is also of the form 1 mod 4.

Thus, there exists at least one prime factor of the form 1 mod 3, which is in Q.

(ii) Suppose on the contrary that there are finitely many primes in Q. Then, $Q = \{q_1, \ldots, q_n\}$ with $q_1 = 3$, etc.

From **(i)**, $N = 4q_2 \dots q_n + 3$ is divisible by some prime in Q. However, none of the q_i's, for $1 \leq i \leq n$, divides N.
Thus, there are infinitely many primes in Q.

Remark for Question 7: The infinitude of primes of the form $4k+3$ is a particular case of Dirichlet's theorem on arithmetic progressions. It states that if $\gcd(a,b) = 1$, then there are infinitely many primes of the form $an+b$.

Question 8

(i) **(a)** By considering the sequence $1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,\dots$, the period is 8

(b) By considering the sequence $1,1,2,3,1,0,1,1,2,3,1,0,1,\dots$, the period is 6

(ii) Modulo m, there are m possible values which are $0,1,2,\dots,m-1$. So, there are m^2 distinct pairs.
As $1 \leq j < k \leq m^2+1$, by considering m^2+1 pairs of (F_i, F_{i+1}) modulo m, where $1 \leq i \leq m^2+1$, the result follows by the pigeonhole principle.

(iii) Here, we would use the method of strong induction. Unlike the conventional method of mathematical induction, strong induction uses more statements in the induction hypothesis.

Let P_n be the proposition that there exists $j,k \in \mathbb{N}$, where $j < k$, such that $F_{j+n} \equiv F_{k+n} \pmod{m}$ for all $n \in \mathbb{Z}_{\geq 0}$.
When $n = 0$, then $F_j \equiv F_k \pmod{m}$.
When $n = 1$, then $F_{j+1} \equiv F_{k+1} \pmod{m}$.
The base cases P_0 and P_1 are true because in **(ii)**, we established that $\left(F_j, F_{j+1}\right) \equiv (F_k, F_{k+1}) \pmod{m}$.

Assume that P_r and P_{r+1} are true for some $r \in \mathbb{Z}_{\geq 0}$. That is,

$$F_{j+r} \equiv F_{k+r} \pmod{m} \text{ and } F_{j+r+1} \equiv F_{k+r+1} \pmod{m}.$$

To show P_{r+2} is true, we need to prove $F_{j+r+2} \equiv F_{k+r+2} \pmod{m}$.
This is true because

$$\begin{aligned} F_{j+r+2} &= F_{j+r+1} + F_{j+r} \quad \text{by definition of Fibonacci sequence} \\ &\equiv F_{k+r+1} + F_{k+r} \pmod{m} \quad \text{by induction hypothesis} \\ &\equiv F_{k+r+2} \pmod{m} \quad \text{by definition of Fibonacci sequence} \end{aligned}$$

Since P_0 and P_1 are true and P_r and P_{r+1} are true imply P_{r+2} is true, then by strong induction, P_n is true for all $n \in \mathbb{Z}_{\geq 0}$.

(iv) By **(iii)**, for any positive integer m, the Fibonacci sequence modulo m is periodic.
Hence, there exists a pair (F_i, F_{i+1}) such that $F_i \equiv F_1 \equiv 1 \pmod{m}$ and $F_{i+1} \equiv F_2 \equiv$

$2 \pmod{m}$.
So, $F_{i-1} \equiv 0 \pmod{m}$.

Remark for Question 8: I found an interesting post on StackExchange which is related to **(iii)**.

CHAPTER 4

2018 Paper Solutions

Question 1

(i)

$$F_n(0) = \sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1},$$

which follows by using partial fractions and the method of difference.

As n tends to infinity, $F_n(0)$ increases and tends to 1.

(ii) (a)

$$\begin{aligned} F_n(x) &= \sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+1} + \frac{2}{(r-1)x+1} - \frac{2}{rx-1}\right] \\ &= \underbrace{\sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+1}\right]}_{\mathrm{F}_n(0)} + 2\sum_{r=1}^{n} \left[\frac{1}{(r-1)x+1} - \frac{1}{rx-1}\right] \\ &= 1 - \frac{1}{n+1} + 2\left(1 - \frac{1}{nx+1}\right) \quad \text{by the method of difference} \\ &= 3 - \frac{1}{n+1} - \frac{2}{nx+1} \end{aligned}$$

(b)

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if } x = 0; \\ 3 & \text{if } x \neq 0 \end{cases}$$

Question 2

(i) Starting with the RHS, let $t = a - x$. When $x = 0$, then $t = a$; when $x = a$, then $t = 0$. Also, $dt = -dx$.

The RHS becomes

$$-\int_0^a f(t)\, dt = \int_0^a f(t)\, dt = \int_0^a f(x)\, dx.$$

(ii) Since f is symmetrical about $x = \frac{1}{2}a$, then $f\left(x + \frac{1}{2}a\right) = f\left(-x + \frac{1}{2}a\right)$.
Replacing x with $x - \frac{1}{2}a$, we have $f(x) = f(a-x)$.
Considering the LHS,

$$\begin{aligned}\int_0^a xf(x)\ dx &= \int_0^a (a-x)\, f(a-x)\ dx\\ &= a\int_0^a f(a-x)\ dx - \int_0^a xf(a-x)\ dx\end{aligned}$$

Using **(i)**, the integrals become

$$a\int_0^a f(x)\,dx - \int_0^a xf(x)\,dx.$$

So,

$$2\int_0^a xf(x)\,dx = a\int_0^a f(x)\,dx.$$

Dividing both sides by 2 yields the result.

(iii) Let

$$g(x) = \frac{x\sin x}{1+\cos^2 x}.$$

Then, g is even because $g(-x) = g(x)$, so the integrand g is symmetrical about $x = 0$.
Setting $a = 0$ in **(ii)**, we have

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx.$$

For the integral on the right, let $u = \cos x$, so $du = -\sin x\,dx$. The integral becomes

$$-\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}\,du = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+u^2}\,du = \frac{\pi}{2}\left[\tan^{-1} u\right]_{-1}^{1} = \frac{\pi^2}{4}.$$

Question 3

(i) Without loss of generality, it suffices to show that

$$\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} \geq 0,$$

where $a + b \geq c$, which is a consequence of the triangle inequality.

The LHS can be written as

$$\frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} = \frac{a(1+b)(1+c) + b(1+a)(1+c) - c(1+a)(1+b)}{(1+a)(1+b)(1+c)}.$$

Since the denominator is always positive and the numerator can be expanded and simplified as $a + b - c + 2ab + abc \geq 0$, the result follows.

(ii) Without a loss of generality, it suffices to show that $\sqrt{a} + \sqrt{b} - \sqrt{c} \geq 0$, where $a + b \geq c$.

Think of $\sqrt{a}+\sqrt{b}-\sqrt{c}$ as the difference of $\sqrt{a}+\sqrt{b}$ and $\sqrt{c}$. By multiplying and dividing by its 'conjugate', the LHS can be written as

$$\sqrt{a}+\sqrt{b}-\sqrt{c}=\frac{\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}{\sqrt{a}+\sqrt{b}+\sqrt{c}}.$$

Similar to **(i)**, we consider the numerator, which can be written as $\left(\sqrt{a}+\sqrt{b}\right)^2-c=a+b-c+2\sqrt{ab}\geq 0$. The result follows.

(iii) Without a loss of generality, it suffices to show that

$$\sqrt{a(b+c-a)}+\sqrt{b(c+a-b)}-\sqrt{c(a+b-c)}\geq 0,$$

where $a+b\geq c$ and a,b,c are the lengths of a triangle.

Let the triangle's perimeter be P, so $P=a+b+c$. So,

$$\sqrt{a(b+c-a)}+\sqrt{b(c+a-b)}-\sqrt{c(a+b-c)}=\sqrt{a(P-2a)}+\sqrt{b(P-2b)}-\sqrt{c(P-2c)}.$$

Let $x=\sqrt{a(P-2a)}$, $y=\sqrt{b(P-2b)}$ and $z=\sqrt{c(P-2c)}$.
Using the cosine rule, say we have a triangle XYZ with $XY=z$, $YZ=x$, $ZX=y$ and $\angle XYZ=\theta$. Then,

$$\begin{aligned}\cos\theta&=\frac{x^2+y^2-z^2}{2xy}\\&=\frac{a(P-2a)+b(P-2b)-c(P-2c)}{2\sqrt{ab(P-2a)(P-2b)}}\\&=\sqrt{\frac{(c-a+b)(c+a-b)}{4ab}}\end{aligned}$$

As $|\cos\theta|\leq 1$, it follows that

$$\begin{aligned}\frac{(c-a+b)(c+a-b)}{4ab}&\leq 1\\c^2&\leq a^2+2ab+b^2\\c&\leq a+b\end{aligned}$$

and the result follows.

Question 4

(i) **(a)** Number of ways is $\binom{7+4-1}{4-1}=120$

(b) The question is equivalent to asking the number of integer solutions to the equation

$$x_1+x_2+x_3+x_4=7, \text{ where all the } x_i\text{'s}\geq 1.$$

Letting $x_i=1-y_i$, we have

$$y_1+y_2+y_3+y_4=3, \text{ where all the } y_i\text{'s}\geq 0.$$

So, the number of ways is $\binom{3+4-1}{3}=20$.

(a) Number of ways is $4^7 = 16384$

(b) Fix any T-shirt in the 1st slot, then the 2nd slot can contain either of the remaining 3 types of T-shirts. Repeating this process up to the 7th slot, we see there are $4\left(3^{7-1}\right) = 2916$ ways.

(c) Let A_i denote the event that the T-shirt of the i^{th} colour is not used, where $1 \le i \le 4$.
So,

$$\begin{aligned}\sum_{i=1}^{4} |A_i| &= \binom{4}{1} 3^7 \\ \sum_{i<j} \left|A_i \cap A_j\right| &= \binom{4}{2} 2^7 \\ \sum_{i<j<k} \left|A_i \cap A_j \cap A_k\right| &= \binom{4}{3} 1^7\end{aligned}$$

By the principle of inclusion and exclusion, the answer is $4^7 - \binom{4}{1}3^7 + \binom{4}{2}2^7 - \binom{4}{3}1^7 = 8400.$

Question 5

(i) **(a)** An $a \times b$ rectangle and a $p \times q$ rectangle have ab and pq squares respectively. Since some number of rectangles are used to tessellate the large board, the result follows.

(b) Suppose the base and height of the rectangle are denoted by a and b respectively. If the large board is tessellated from left to right with α vertical and β horizontal rectangles, then the bottom row of the board has $\alpha a + \beta b$ squares. As each row has q squares, then $q = \alpha a + \beta b$. Similarly, if we tessellate the board from the bottom to the top, we have $p = \gamma a + \delta b$. Since $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}$, the result follows.

(c) In each $a \times b$ tile, along each row, there is only one shaded square. Since there are b rows, there will be b shaded squares in each tile. If k tiles are used in the tessellation, there will be kb shaded squares on the large board. From **(a)**, as ab is a factor of pq, then there exists $k \in \mathbb{N}$ such that $kab = pq$. Hence, kb refers to the number of shaded squares on the board.

(ii) **(a)** Given that

$$\begin{aligned}p &\equiv r \pmod{a},\ 0 \le r < a \\ q &\equiv s \pmod{a},\ 0 \le s < a\end{aligned}$$

then there exists $m, n \in \mathbb{N}$ such that

$$p = ma + r \text{ and } q = na + s.$$

Consider a large $p \times q$ rectangle. So, $pq = a^2mn + ams + anr + rs$. We remove an $r \times s$ rectangle from the bottom-right corner so the remaining figure has

$$\frac{pq - rs}{a} = mna + ms + nr$$

non-overlapping rows or columns of $a \times 1$ rectangles. So, this figure comprises $\frac{pq-rs}{a}$ shaded blocks.
As $t = \min\{r, s\}$ and there are t shaded blocks in the $r \times s$ rectangle, the result follows.

(b) From **(iia)**, the number of shaded squares in the $p \times q$ rectangle is $\frac{pq-rs}{a} + t$.
From **(ic)**, the number of shaded squares in the tessellated $p \times q$ rectangle is $\frac{pq}{a}$.
Equating the two, we have $at = rs$.
If $t = r$, then $r(a - s) = 0$. However, if $r \neq 0$, then $s = a$, which is a contradiction as $s < a$. So, $r = 0$ and $a|p$.
If $t = s$, then $s(a - r) = 0$, which implies that $s = 0$ and so $a|q$.

Question 6

(a) Let $A = \{a_1, \ldots, a_n\}$ be a group of n students, and for each i, the number of students a_i knows is $f(i)$. Also, let $B = \{1, \ldots, n\}$. Then, for all $1 \leq i \leq n$, we have $0 \leq f(i) \leq n - 1$.

- **Case 1:** Suppose there exist $i, j \in B$, where $i \neq j$, such that $f(i) = f(j) = 0$. Then, a_i and a_j both have no friends, so the result is trivial.
- **Case 2:** Suppose there is precisely one $i \in [1, n]$ such that $f(i) = 0$. Then, for all $j \in [1, n] \setminus \{i\}$, we have $1 \leq f(j) \leq n - 2$. By the pigeonhole principle, we have $j, k \in [1, n] \setminus \{i\}$, where $j \neq k$, such that $f(j) = f(k)$.
- **Case 3:** Suppose $f(i) > 0$ for all $1 \leq i \leq n$. Then, we have $1 \leq f(i) \leq n - 1$. By the pigeonhole principle, for $1 \leq i, j \leq n$, where $i \neq j$, we have $f(i) = f(j)$.

We assume that friendship is a symmetric relation, meaning if a_1 is a friend of a_2, then a_2 is also a friend of a_1.

(b) Define the fractional part of x, $\{x\}$, to be $x - \lfloor x \rfloor$.
Consider $\{kx\}$, where $1 \leq k \leq n$, and subintervals of $[0, 1)$ each of length $\frac{1}{n}$. These are

$$I_1 = \left[0, \frac{1}{n}\right),\ I_2 = \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, I_n = \left[\frac{n-1}{n}, 1\right).$$

- **Case 1:** Suppose some $\{kx\}$ falls in I_1. As $\{kx\} < \frac{1}{n}$, then

$$kx - \lfloor kx \rfloor < \frac{1}{n}$$
$$\left| x - \frac{\lfloor kx \rfloor}{k} \right| < \frac{1}{kn}$$

so by setting $a = \lfloor kx \rfloor$ and $b = k$, we establish the desired inequality.

- **Case 2:** Suppose none of the $\{kx\}$ falls in I_1. By the pigeonhole principle, at least two $\{kx\}$ fall in the same I_i, where $2 \leq i \leq n$. Let

$$\frac{i-1}{n} \leq \{px\} < \frac{i}{n} \text{ and } \frac{i-1}{n} \leq \{qx\} < \frac{i}{n}.$$

Then,

$$\begin{aligned} |\{px\} - \{qx\}| &< \frac{1}{n} \\ |px - \lfloor px \rfloor - qx + \lfloor qx \rfloor| &< \frac{1}{n} \\ |(p-q)x - (\lfloor px \rfloor - \lfloor qx \rfloor)| &< \frac{1}{n} \\ \left| x - \frac{\lfloor px \rfloor - \lfloor qx \rfloor}{p-q} \right| &< \frac{1}{(p-q)n} \end{aligned}$$

so by setting $a = \lfloor px \rfloor - \lfloor qx \rfloor$ and $b = p - q$, we establish the desired inequality.

Remark for Question 6: For **(b)**, a faster method without considering the pigeonhole principle is as such. We start off by noting that $nx - \lfloor nx \rfloor = \{nx\}$, so

$$\begin{aligned} x - \frac{\lfloor nx \rfloor}{n} &= \frac{\{nx\}}{n} \\ \left| x - \frac{\lfloor nx \rfloor}{n} \right| &= \left| \frac{\{nx\}}{n} \right| < \frac{1}{n} \end{aligned}$$

As $a, b \in \mathbb{Z}$ and $1 \leq b \leq n$, we can set $a = \lfloor nx \rfloor$ and $b = 1$ and the result follows.

The interested can look up Diophantine approximation.

Question 7

(i) Using the substitution $t = \frac{dy}{dx}$, we have $\frac{dt}{dx} = \frac{d^2y}{dx^2}$.
Differentiating 1 with respect to x, we have

$$\begin{aligned} y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 &= 2x\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 \\ y &= 2x\frac{dy}{dx} \end{aligned}$$

Solving the differential equation yields $\frac{1}{2}\ln|x| = \ln|y| + c$.
So $y^2 = Ax$, where $A = e^{-2c}$. Thus, c and A are constants.

Differentiating $y^2 = Ax$ with respect to x, we have

$$2y\frac{dy}{dx} = A,$$

and substituting this into the original differential equation, we have $A = 4$.
Hence, the equation of S is $y^2 = 4x$.

(ii) First, we show that if a straight line is tangent to S, then it is a solution to equation 1.
Note that
$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$
Suppose the line is tangent to the curve at $P\left(\frac{1}{4}a^2, a\right)$. Then, the equation of the tangent at P is
$$y - a = \frac{2}{a}\left(x - \frac{a^2}{4}\right)$$
$$y = \frac{2}{a}x + \frac{a}{2}$$
Substituting this into the original differential equation, the result follows.

Next, we show that if a straight line is a solution to 1, then it is tangent to S.
Note that any line satisfies $y = mx + c$. Substituting this into the original differential equation yields $m(mx+c) = xm^2 + 1$. Since this holds for any $x \in \mathbb{R}$, then $c = \frac{1}{m}$.
The equation of the line becomes
$$y = mx + \frac{1}{m}.$$
Note that $2y\frac{dy}{dx} = 4$ and as $\frac{dy}{dx} = m$, then $y = \frac{2}{m}$. As such, $x = \frac{1}{m^2}$.
Since $y^2 = 4x$, then
$$\left(mx + \frac{1}{m}\right)^2 = 4x$$
$$m^2x^2 - 2x + \frac{1}{m^2} = 0$$
The discriminant of the above quadratic equation is 0, implying that $y = mx + \frac{1}{m}$ is tangent to the curve.
In particular, the line is tangential at the point $\left(\frac{1}{m^2}, \frac{2}{m}\right)$.

Question 8

(i)
$$\sum_{r=1}^{3} n\left(\frac{11}{7}r\right) = n\left(\frac{11}{7}\right) + n\left(\frac{22}{7}\right) + n\left(\frac{33}{7}\right)$$
$$= 2 + 3 + 5 = 10$$

(ii) We have $\sum_{r=1}^{5} n\left(\frac{7}{11}r\right) = 1 + 1 + 2 + 3 + 3 = 10.$

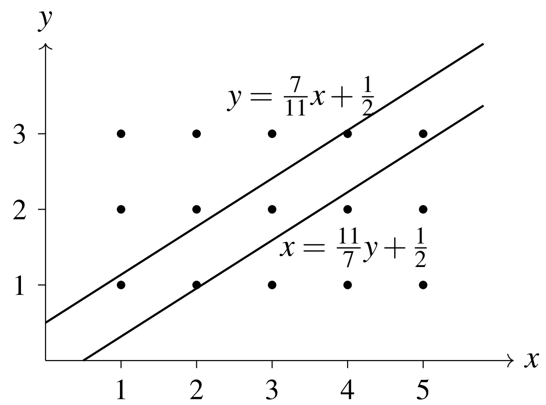
From the table, we see that there are also 10 points underneath the line $y = \frac{7}{11}x + \frac{1}{2}$.

Line	Lattice points underneath $y=\frac{7}{11}x+\frac{1}{2}$	Number of lattice points
$x=1$	$(1,1)$	1
$x=2$	$(2,1)$	1
$x=3$	$(3,1)$ and $(3,2)$	2
$x=4$	$(4,1)$, $(4,2)$ and $(4,3)$	3
$x=5$	$(5,1)$, $(5,2)$ and $(5,3)$	3

(iii) Suppose $\left(0,\frac{1}{2}\right)$ is mapped to (a,b) by the rotation. Since $(3,2)$ is the midpoint of $\left(0,\frac{1}{2}\right)$ and (a,b), by the midpoint formula, $a=6$ and $b=\frac{7}{2}$.
Hence, $\left(6,\frac{7}{2}\right)$ lies on the rotated line.

Suppose $\left(-\frac{11}{14},0\right)$ is mapped to (c,d) by the rotation. In a similar fashion, $c=\frac{95}{14}$ and $d=4$.
Hence, $\left(\frac{95}{14},4\right)$ lies on the rotated line.

The equation of the line joining $\left(6,\frac{7}{2}\right)$ and $\left(\frac{95}{14},4\right)$ is $x=\frac{11}{7}y+\frac{1}{2}$.



As a rotation by 180° about $(3,2)$ leaves all the lattice points unchanged, then by symmetry,

$$\sum_{r=1}^{3} n\left(\frac{11}{7}r\right)$$

denotes the total number of lattice points to the left of the line $x=\frac{11}{7}y+\frac{1}{2}$ for $y=1,2,3$, which is equal to the total number of lattice points underneath the line $y=\frac{7}{11}x+\frac{1}{2}$ for $x=1,2,3,4,5$. The latter is equal to

$$\sum_{r=1}^{5} n\left(\frac{7}{11}r\right).$$

(iv) Note that

$$\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$$

denotes the number of integer points in the rectangle bounded by $1\le x\le \frac{1}{2}(p-1)$, $1\le y\le \frac{1}{2}(q-1)$.
Also, $n\left(\frac{q}{p}r\right)$ denotes the number of integer points underneath the line $y=\frac{q}{p}x+\frac{1}{2}$

for $x = r$.

Then,

$$\sum_{r=1}^{\frac{p-1}{2}} n\left(\frac{q}{p}r\right) = \text{total number of integer points underneath the line } y = \frac{q}{p}x + \frac{1}{2} \text{ for } 1 \leq x \leq \frac{p-1}{2}$$

and

$$\sum_{r=1}^{\frac{q-1}{2}} n\left(\frac{p}{q}r\right) = \text{total number of integer points to the left of the line } x = \frac{p}{q}y + \frac{1}{2} \text{ for } 1 \leq y \leq \frac{q-1}{2}.$$

Let A and B be the set of integer points underneath the line $y = \frac{q}{p}x + \frac{1}{2}$ and to the left of $x = \frac{p}{q}y + \frac{1}{2}$ respectively.

Thus,

$$|A| = \sum_{r=1}^{\frac{p-1}{2}} n\left(\frac{q}{p}r\right) \text{ and } |B| = \sum_{r=1}^{\frac{q-1}{2}} n\left(\frac{p}{q}r\right).$$

As such,

$$\begin{aligned} N &= |A \cap B| \\ &= |A| + |B| - |A \cup B| \\ &= 2|A| - \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) \end{aligned}$$

and therefore, $N + \left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right) \equiv 0 \pmod 2$.

Remark for Question 8: For **(iii)**, those who have background knowledge of linear algebra would find the use of the rotation matrix extremely helpful in the early part of this question. Overall, this question deals with a geometric proof of the law of quadratic reciprocity, which was established by Gotthold Eisenstein.

CHAPTER 5

2019 Paper Solutions

Question 1

(i) By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(x^2+y^2+z^2\right)\left(2^2+3^2+6^2\right) &\geq (2x+3y+6z)^2\\ (2x+3y+6z)^2 &\leq 49\left(x^2+y^2+z^2\right)\\ 2x+3y+6z &\leq 7 \quad \text{since } x^2+y^2+z^2=1\end{aligned}$$

(ii) From **(i)**, by setting up the inequality and noting that $2^2+3^2+6^2=7^2$, we have $x=\frac{2}{7}$, $y=\frac{3}{7}$ and $z=\frac{6}{7}$.

(iii) By the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.$$

Since $\sum_{i=1}^{n} x_i^2 = 1$, we have $n \geq \left(\sum_{i=1}^{n} x_i\right)^2$ so the required maximum value is $\sqrt{n}$.

(iv) Let the length of each square be l_i, where $i \geq 1$ and suppose there are n squares. Since $\sum_{i=1}^{n} 4l_i = 18$, then $\sum_{i=1}^{n} l_i = \dfrac{9}{2}$.

Also, the area of the large unit square is 1 so $\sum_{i=1}^{n} l_i^2 = 1$.

By the Cauchy-Schwarz inequality, $n \geq \frac{81}{4}$ so there are more than 20 such squares.

Question 2

(i) **(a)** Number of ways is $\dbinom{8+4-1}{4-1} = 165$

(b) The question is equivalent to asking the number of integer solutions to the equation

$$x_1+x_2+x_3+x_4=8, \text{ where all the } x_i\text{'s} \geq 1.$$

Letting $x_i = 1 - y_i$, we have

$$y_1 + y_2 + y_3 + y_4 = 4, \text{ where all the } y_i\text{'s} \geq 0.$$

So, the number of ways is $\binom{4+4-1}{3} = 35$

(ii) (a) Number of ways is $4^8 = 65536$

(b) Number of ways is $4 \times 3^7 = 8748$

(c) We assign each base to an arbitrary index i, where $1 \leq i \leq 4$. Let A_i denote the event that base i is not used.
So,

$$\sum_{i=1}^{4} |A_i| = \binom{4}{1} 3^8$$

$$\sum_{i<j} |A_i \cap A_j| = \binom{4}{2} 2^8$$

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k| = \binom{4}{3} 1^8$$

By the principle of inclusion and exclusion, the answer is $4^8 - \binom{4}{1} 3^8 + \binom{4}{2} 2^8 - \binom{4}{3} 1^8 = 40824$.

Remark for Question 2: This question is very similar to Question 4 of the 2018 A-Level paper.

Question 3

(i) (a) Let P_i be the proposition that $x_i \geq \frac{1}{i}$ for all $i \in \mathbb{N}$.
By definition, $x_1 = 1$ so P_1 is true.
Assume P_k is true for some $k \in \mathbb{N}$. That is, $x_k \geq \frac{1}{k}$.
To prove P_{k+1} is true, we need to show that $x_k \geq \frac{1}{k+1}$.
So,

$$\begin{aligned} x_{k+1} &= \frac{k+a}{k+1} x_k \quad \text{by definition of the recurrence relation} \\ &\geq \frac{k+a}{k(k+1)} \quad \text{by induction hypothesis} \\ &= \frac{1}{k+1}\left(1 + \frac{a}{k}\right) \\ &\geq \frac{1}{k+1} \end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_i is true for all $i \in \mathbb{N}$.

(b)

$$\begin{aligned}\sum_{i=n+1}^{2n} x_i &\geq \sum_{i=n+1}^{2n} \frac{1}{i} \quad \text{by } \textbf{(i)} \\ &\geq \sum_{i=n+1}^{2n} \frac{1}{2n} \quad \text{since } i \leq 2n \\ &= \frac{1}{2}\end{aligned}$$

(c) Suppose on the contrary that $\sum_{i=1}^{\infty} x_i$ is bounded.

Note that $\sum_{i=1}^{n} x_i$ is strictly increasing and bounded above. By the monotone convergence theorem, it converges to some finite number, say N.

From **(b)**, we established that

$$\sum_{i=n+1}^{2n} x_i \geq \frac{1}{2}$$

so

$$\sum_{i=1}^{2n} x_i \geq \frac{1}{2} + \sum_{i=1}^{n} x_i.$$

As $n \to \infty$, we have

$$\begin{aligned}\sum_{i=1}^{\infty} x_i &\geq \frac{1}{2} + \sum_{i=1}^{\infty} x_i \\ N &\geq \frac{1}{2} + N\end{aligned}$$

which is a contradiction.

(ii) **(a)** The recurrence relation can be written as $(i+1)x_{i+1} = (i+a)x_i$ so

$$\begin{aligned}(i+1)x_{i+1} - ix_i &= ax_i \\ \sum_{i=m}^{n} [(i+1)x_{i+1} - ix_i] &= \sum_{i=m}^{n} ax_i \\ a\sum_{i=m}^{n} x_i &= (n+1)x_{n+1} - mx_m \quad \text{by the method of difference}\end{aligned}$$

(b) Let $b = -a > 0$. Note that

$$\begin{aligned}\frac{i-b}{i+1} &< 0 \quad \text{if } i < b; \\ &> 0 \quad \text{if } i > b\end{aligned}$$

Hence, the non-zero terms of the sequence x_i alternates in signs for $1 \leq i \leq \lfloor b \rfloor$.

If $x_{\lfloor b \rfloor} < 0$, then $x_{\lfloor b \rfloor+1} \geq 0$, $x_{\lfloor b \rfloor+2} \geq 0$, and so on. Hence, for all $k \geq 1$, $x_{\lfloor b \rfloor+k} \geq 0$.

Similarly, for all $k \geq 1$, if $x_{\lfloor b \rfloor} > 0$, then $x_{\lfloor b \rfloor+k} \leq 0$.

If $b \in \mathbb{N}$, for all $k \geq 1$, then $x_{b+k} = 0$.

Hence, for sufficiently large $m, n \in \mathbb{N}$, in particular $n > m > \lfloor b \rfloor$, x_m and x_n will have the same sign and the result follows.

Question 4

(i) **(a)** Consider the n-digit number having 1st digit 2. There are Y_n such numbers. Then, consider the $(n-1)$-digit number from the 2nd to the last digit. The 2nd digit has to be 1 or 3.

Define Z_n to be the number of n-digit numbers with first digit 3. By symmetry, $Y_n = Z_n$.

- **Case 1:** If the 2nd digit is 1, there are X_{n-1} $(n-1)$-digit numbers.
- **Case 2:** If the 2nd digit is 3, there are Z_{n-1} $(n-1)$-digit numbers. As $Y_{n-1} = Z_{n-1}$, there are Y_{n-1} $(n-1)$-digit numbers.

Since the two cases are mutually exclusive, then $Y_n = X_{n-1} + Y_{n-1}$.

(b) Consider the n-digit number having 1st digit 1. There are X_n such numbers. Consider the $(n-1)$-digit number from the 2nd to the last digit. There is no restriction on the 2nd digit.

- **Case 1:** If the 2nd digit is 1, there are X_{n-1} $(n-1)$-digit numbers.
- **Case 2:** If the 2nd digit is 2, there are Y_{n-1} $(n-1)$-digit numbers.
- **Case 3:** If the 2nd digit is 3, there are Z_{n-1} $(n-1)$-digit numbers. As $Y_{n-1} = Z_{n-1}$, then there are Y_{n-1} $(n-1)$-digit numbers.

Since the three cases are mutually exclusive, then $X_n = X_{n-1} + 2Y_{n-1}$.

(c) We have

$$\begin{aligned}
X_{n+1} &= X_n + 2Y_n \quad \text{by } \textbf{(b)} \\
&= X_n + 2\left(X_{n-1} + Y_{n-1}\right) \quad \text{by } \textbf{(a)} \\
&= X_n + 2X_{n-1} + X_n - X_{n-1} \quad \text{by } \textbf{(b)} \\
&= 2X_n + X_{n-1}
\end{aligned}$$

(ii) Let P_n be the proposition that $X_n \equiv n^2 - n + 1 \pmod 4$ for all $n \in \mathbb{N}$.
When $n = 1$, we have $X_1 = 1 \equiv 1 \pmod 4$ so P_1 is true.
When $n = 2$, we have $X_2 = 3 \equiv 3 \pmod 4$ so P_2 is true.

Assume P_{k-1} and P_k are true for some $k \in \mathbb{N}$, where $k \geq 2$. That is,

$$X_{k-1} \equiv (k-1)^2 - (k-1) + 1 \pmod 4 \quad \text{and} \quad X_k \equiv k^2 - k + 1 \pmod 4$$

respectively.

To show that P_{k+1} is true, we need to prove that $X_{k=1} \equiv (k+1)^2 - (k+1) + 1 \pmod 4$.
Note that $(k+1)^2 - (k+1) + 1 = k^2 + k + 1$.

Using the recurrence relation,

$$\begin{aligned} X_{k+1} &= 2X_k + X_{k-1} \\ &\equiv 2\left(k^2-k+1\right) + (k-1)^2 - (k-1) + 1 \pmod 4 \\ &= 2k^2 - 2k + 2 + k^2 - 2k + 1 - k + 2 \\ &= 3k^2 - 5k + 5 \\ &\equiv k^2 + k + 1 + 2(k-1)(k-2) \pmod 4 \end{aligned}$$

As $k-1$ and $k-2$ are of opposite parities, it implies that $(k-1)(k-2)$ is even so $X_{k+1} \equiv k^2+k+1 \pmod 4$.

Since P_1 and P_2 are true and P_{k-1} and P_k are true imply that P_{k+1} is true, by strong induction, P_n is true for all $n \in \mathbb{N}$.

(iii)

$$\begin{aligned} T_n &= X_n + Y_n + Z_n \\ &= X_n + 2Y_n \\ &= X_{n+1} \quad \text{by } \textbf{(ib)} \\ &\equiv n^2 + n + 1 \pmod 4 \end{aligned}$$

Question 5

(i) Using the substitution, we have

$$\frac{dt}{dx} = \frac{d^2u}{dx^2}.$$

The differential equation becomes $\frac{dt}{dx} = t$.
So, $\int \frac{1}{t}\,dt = \int dx$, which implies $\ln|t| = x + c$, where c is a constant. So, $t = Ae^x$, where $A = e^c$.
Since $\frac{du}{dx} = Ae^x$, then $\int du = \int Ae^x\,dx$, implying that $u = Ae^x + k$, where k is a constant too.

(ii) Let $u = e^{-\int f(x)y\,dx}$.
Then,

$$\begin{aligned} \frac{du}{dx} &= -e^{-\int f(x)y\,dx} f(x)\,y \\ &= -uyf(x) \end{aligned}$$

Differentiating one more time yields

$$\begin{aligned} \frac{d^2u}{dx^2} &= -uyf'(x) + f(x)\left(-u\frac{dy}{dx} - y\frac{du}{dx}\right) \\ &= -uyf'(x) - f(x)\left(u\frac{dy}{dx} + y\frac{du}{dx}\right) \\ &= -uyf'(x) - uf(x)\frac{dy}{dx} - yf(x)\frac{du}{dx} \end{aligned}$$

Rearranging,

$$uf(x)\frac{dy}{dx} = -uyf'(x) - yf(x)\frac{du}{dx} - \frac{d^2u}{dx^2}.$$

As $\dfrac{dy}{dx} = f(x)y^2 + g(x)y$, then

$$\begin{aligned}
uy^2[f(x)]^2 + uyf(x)g(x) + uyf'(x) + yf(x)\frac{du}{dx} + \frac{d^2u}{dx^2} &= 0 \\
uy^2[f(x)]^2 + uyf(x)g(x) + uyf'(x) + yf(x)[-uyf(x)] + \frac{d^2u}{dx^2} &= 0 \\
\frac{d^2u}{dx^2} + uyf(x)g(x) + uyf'(x) &= 0 \\
\frac{d^2u}{dx^2} - g(x)\frac{du}{dx} + uyf'(x) &= 0 \\
f(x)\frac{d^2u}{dx^2} - f(x)g(x)\frac{du}{dx} + uyf(x)f'(x) &= 0 \\
f(x)\frac{d^2u}{dx^2} - \left[f'(x) + f(x)g(x)\right]\frac{du}{dx} &= 0
\end{aligned}$$

(iii) As $\dfrac{dy}{dx} = e^{-2x}y^2 + 3y$, then $f(x) = e^{-2x}$ and $g(x) = 3$.
Using **(ii)**,

$$\begin{aligned}
f(x)\frac{d^2u}{dx^2} - \left[f'(x) + f(x)g(x)\right]\frac{du}{dx} &= 0 \\
e^{-2x}\frac{d^2u}{dx^2} - e^{-2x}\frac{du}{dx} &= 0
\end{aligned}$$

As e^{-2x} is non-zero for all $x \in \mathbb{R}$, then $\frac{d^2u}{dx^2} = \frac{du}{dx}$.
From **(i)**, the solution is of the form $u = e^{x+c} + k$ for constants c and k. So,

$$\begin{aligned}
e^{-\int e^{-2x}y\,dx} &= e^{x+c} + k \\
-\int e^{-2x}y\,dx &= \ln\left(e^{x+c} + k\right) \\
-e^{-2x}y &= \frac{e^{x+c}}{e^{x+c} + k} \\
y &= -\frac{e^{3x+c}}{e^{x+c} + k}
\end{aligned}$$

When $x = 0$, $y = -\frac{1}{4}$, so $k = 3e^c$. Therefore, $y = -\frac{e^{3x}}{e^x+3}$.

Question 6

(i) Let $x_1, x_2, \ldots, x_{2n}$ denote the positions of the $(+1)$'s and (-1)'s and each $+1$ precedes a corresponding -1. Denote x_i, where $1 \le i \le 2n$ to be the starting point.

There exists $i \in [1, 2n]$ such that $(x_i, x_{i+1}) = (+1, -1)$ or $(-1, +1)$, meaning there are two adjacent points of opposite polarity. Delete x_i and x_{i+1}, so we would have

$2n-2$ positions remaining. Repeat this process until we have 2 points remaining, say x_j and x_k. As such, $(x_j, x_k) = (+1, -1)$.

Suppose x_j is the final position of the $+1$. Restoring all the positions and moving in a clockwise manner, the next position must be either $+1$ or -1. If we proceed with the former, then $T_i = 2$. For the latter, $T_i = 0$. Subsequently, the next position must be either -1 or $+1$ respectively and repeating this process, we conclude that there does not exist $i \in [1, 2n]$ such that $T_i < 0$.

(ii) Regardless of the polarity of the first position, $T_1 \equiv 1 \pmod 2$. As i increases by 1, then the polarity of T_i changes. If n is odd, then $T_i + T_{i+1}$ is odd so

$$n + \sum_{i=1}^{2n} T_i = 2\lambda + 1 + 2\mu + 1 \equiv 0 \pmod 2.$$

If n is even, then $T_i + T_{i+1}$ is even so

$$n + \sum_{i=1}^{2n} T_i = 2\lambda + 2\mu \equiv 0 \pmod 2.$$

Question 7

(i) $c\cos\theta + d\sin\theta < a$ and $c\sin\theta + d\cos\theta < b$

(ii) We first prove the forward direction by contraposition. Suppose $d \geq b$. Then, $a > c \geq d \geq b$ and

$$\begin{aligned} c\sin\theta + d\cos\theta &\geq b(\sin\theta + \cos\theta) \\ &= b\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) \end{aligned}$$

As $0 < \theta < \frac{\pi}{2}$, then $\cos\left(\theta - \frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$, so $b\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) > b$, which completes the proof.

Next, we prove the backward direction. Choose θ sufficiently small such that

$$c\sin\theta < \varepsilon = \min\{a-c, b-d\}.$$

Then,

$$c\cos\theta + d\sin\theta \leq c\cos\theta + c\sin\theta < c + \varepsilon \leq a \quad \text{and} \quad d\cos\theta + c\sin\theta < d + \varepsilon \leq b,$$

which completes the proof.

(iii) Let θ_0 be the angle for which the $c \times d$ rectangle is strictly contained in the $a \times b$ rectangle.
By **(i)**, we must have

$$c\cos\theta_0 + d\sin\theta_0 < a \quad \text{and} \quad c\sin\theta_0 + d\cos\theta_0 < b.$$

From **(ii)**, by considering $c\cos\theta + d\sin\theta \le c\cos\theta + c\sin\theta < c + \varepsilon \le a$ and substituting it into the above inequalities, we have

$$c\sin\left(\frac{\pi}{2} - \theta_0\right) + d\cos\left(\frac{\pi}{2} - \theta_0\right) < b \le a.$$

Let $f(\theta) = c\cos\theta + d\sin\theta$, where $0 \le \theta \le \frac{\pi}{2}$, $\theta_1 = \min\left\{\theta_0, \frac{\pi}{2} - \theta_0\right\}$ and $\theta_2 = \max\left\{\theta_0, \frac{\pi}{2} - \theta_0\right\}$.
Then, $f(\theta_1), f(\theta_2) < a$ and $\theta_1 \le \frac{\pi}{4} \le \theta_2$.
Since $f(\theta) > 0$ and $f''(\theta) = -(c\cos\theta + d\sin\theta) < 0$ for $0 \le \theta \le \frac{\pi}{2}$, then f attains a maximum at θ_{max}, where

$$f'(\theta_{\max}) = -c\sin\theta_{\max} + d\cos\theta_{\max} = 0.$$

This implies that

$$\theta_{\max} = \tan^{-1}\left(\frac{d}{c}\right) \le \frac{\pi}{4}.$$

Since $f(0) = c \ge a$ and f is increasing on $[0, \theta_{\max}]$, then $\theta_{\max} < \theta_1$. Moreover, as f is decreasing on $\left[\theta_{\max}, \frac{\pi}{2}\right]$, it is also increasing on $[\theta_1, \theta_2]$.

(iv) The condition is that $a > c$ or $a\sqrt{2} > c + d$.

First, we will prove the necessary statement. Suppose a $c \times d$ rectangle can be strictly contained in an $a \times a$ square. If $a > c$, we are done. Otherwise, if $a \le c$, then by **(iii)**, we have $\sqrt{2} > c + d$.

Next, we will prove the sufficiency statement. If $a > c$, by **(ii)**, a $c \times d$ rectangle can be strictly contained in an $a \times a$ square if and only if $a > d$. However, as $a > c \ge d$, then the rectangle can always be contained in the square.

If $\sqrt{2} > c + d$, by considering the inequalities in **(i)** which are

$$c\cos\theta + d\sin\theta < a \quad \text{and} \quad c\sin\theta + d\cos\theta < b,$$

setting $\theta = \frac{\pi}{4}$ into each both yield $\dfrac{c+d}{\sqrt{2}} < a$.

(i) For any $x \in \mathbb{N}$, we have $(12 - x)^2 \equiv x^2 \pmod{12}$ so we only consider the first 6 non-negative square numbers.

$$\begin{aligned}
0^2 &\equiv 0 \pmod{12}\\
1^2 &\equiv 1 \pmod{12}\\
2^2 &\equiv 4 \pmod{12}\\
3^2 &\equiv 9 \pmod{12}\\
4^2 &\equiv 4 \pmod{12}\\
5^2 &\equiv 1 \pmod{12}
\end{aligned}$$

and the result follows.

(ii) Let $N = 9$. We have $5^2 \equiv 7 \pmod 9$ and $7 \in S(9)$ but 7 is non-square.

(iii) For all $N \in \mathbb{N}$, $n[S(N)]$ is greater than or equal to the number of distinct non-negative integers m satisfying $m^2 < N$.

- **Case 1:** Suppose N is non-square. Here, $\sqrt{N} < \lfloor\sqrt{N}\rfloor \notin \mathbb{N}$ so $m < \sqrt{N}$. This implies $0 \le m \le \lfloor\sqrt{N}\rfloor$ so there are $1 + \lfloor\sqrt{N}\rfloor$ distinct non-negative integers m satisfying $m < \sqrt{N}$. Thus, there are at least $1 + \lfloor\sqrt{N}\rfloor$ distinct elements of $S(N)$ and the result follows.
- **Case 2:** Suppose N is square. Here, $\lfloor\sqrt{N}\rfloor = \sqrt{N} \in \mathbb{N}$. This implies that $m \le \sqrt{N}$, so $0 \le m \le \sqrt{N} - 1$. There are $\lfloor\sqrt{N}\rfloor$ distinct non-negative integers m satisfying $m < \sqrt{N}$. Thus, there are at least $\lfloor\sqrt{N}\rfloor$ distinct elements of $S(N)$.

(iv) We consider 2 cases – when λ is even and when λ is odd.

- **Case 1:** If λ is even, then $\frac{\lambda}{2}$ would still be an integer. Thus,

$$x^2 = 17 + 2^{n+1}\left(\frac{\lambda}{2}\right)$$

  and in modulo 2^{n+1}, we have $x^2 \equiv 17 \pmod{2^{n+1}}$ and the result follows.
- **Case 2:** If λ is odd, let $\mu \in \mathbb{Z}$ such that

$$\mu = \frac{\lambda + x + 2^{n-2}}{2}$$

  and the above equation is valid since λ and x are odd, so their sum would be even.

  Hence, $2^{n+1}\mu + 17 = (x + 2^{n-1})^2$ and so $\left(x + 2^{n-1}\right)^2 \equiv 17 \pmod{2^{n+1}}$.

(v) From **(iii)**, $S(2^n)$ has at least $\sqrt{2^n}$ elements.
From **(iv)**, there exist $x, \lambda \in \mathbb{Z}$ such that $x^2 = 17 + 2^n$ for $n \ge 5$.
Note that all squares p^2, where $0 \le p^2 \le 2^n$, are elements of $S(2^n)$ and there are at least $\sqrt{2^n}$ of these elements.
Moreover, from **(iv)**, $17 \in S(2^n)$ and 17 is non-square, and the result follows.

CHAPTER 6

2020 Paper Solutions

Question 1

(i) Observe that on the LHS of the inequality, there are $n-1$ copies of x and 1 copy of y. It is most plausible to apply the AM-GM inequality.
Hence,

$$\frac{(n-1)x+y}{n} \geq \sqrt[n]{x^{n-1}y}.$$

Multiplying both sides by n and raising them to the n^{th} power yields the desired result.
Equality holds if and only if $x = y$.

(ii) For the term $(1+a)^2$, it hints that $a = y$.
Comparing with **(i)**, we have

$$[(n-1)x+a]^n \geq n^n x^{n-1} a.$$

Observe that the power on the LHS must be 2, so $n = 2$. Consequently, $x = 1$.
Thus,

$$(1+a)^2 \geq 4a.$$

Next, for the term $(1+b)^3$, it hints that $b = y$.
Comparing it with **(i)**, we have

$$[(n-1)x+b]^n \geq n^n x^{n-1} b.$$

Observe that the power on the LHS is 3, so $n = 3$. Consequently, $x = \frac{1}{2}$.
Thus,

$$(1+b)^3 \geq \frac{27}{4} b.$$

Lastly, in a similar fashion, one can show that

$$(1+c)^4 \geq \frac{256}{27} c.$$

In each scenario, for equality to be obtained, we must have $a = 1$, $b = \frac{1}{2}$, and $c = \frac{1}{3}$ by the AM-GM inequality.
Multiplying the inequalities

$$(1+a)^2 \geq 4a,\ (1+b)^3 \geq \frac{27}{4}b \text{ and } (1+c)^4 \geq \frac{256}{27}c$$

yields

$$(1+a)^2(1+b)^3(1+c)^4 \geq (4a)\left(\frac{27}{4}b\right)\left(\frac{256}{27}c\right) = 256.$$

However, the original inequality in **(ii)** is strict and $abc = 1$ by the constraint in the question. Previously, we mentioned that $abc = \frac{1}{6}$ by the AM-GM inequality. This contradiction implies that the inequality is strict.

Question 2

(i) Let $y = \frac{1}{ax+b}$. Then, $x = \frac{1}{a}\left(\frac{1-by}{y}\right)$, which implies that $f^{-1}(x) = \frac{1}{a}\left(\frac{1-bx}{x}\right)$, where $x \neq 0$.

(ii) Let $p \in \mathbb{R}$ be arbitrary.
Clearly, p, $f(p)$ and $f^2(p)$ are all not equal to $-\frac{b}{a}$ so $f^3(p)$ exists.
Moreover,

$$\begin{aligned} p &= f^3(p) \\ &= f(f^2(p)) \\ &= \frac{1}{af^2(p)+b} \end{aligned}$$

which is non-zero.
Thus,

$$\begin{aligned} f^2(p) &= f^{-1}(p) \\ \frac{1}{a\left(\frac{1}{ap+b}\right)+b} &= \frac{1-bp}{ap} \\ ap(ap+b) &= (1-bp)\left(a+abp+b^2\right) \\ \left(a+b^2\right)\left(1-bp-ap^2\right) &= 0 \end{aligned}$$

We now consider two cases.

- **Case 1:** Suppose $a = -b^2$. Then,

$$\begin{aligned} f(x) &= \frac{1}{b(1-bx)} \\ f^2(x) &= f\left[\frac{1}{b(1-bx)}\right] \\ &= \frac{bx-1}{b^2x} \\ f^3(x) &= f\left(\frac{bx-1}{b^2x}\right) \\ &= x \end{aligned}$$

Since x was arbitrary, then f^3 fixes all x for which f^3 exists.

- **Case 2:** Suppose $1-bp-ap^2=0$. Then, $\frac{1}{ap+b}=p$, which implies that $f(p)=p$. As such, p is a fixed point of f.

(iii) Note that

$$x_{n+1}=\frac{1}{Ax_n+B}.$$

We have

$$f(x_n)=\frac{1}{Ax_n+B}, \quad \text{where } x_n \neq -\frac{B}{A}.$$

Setting $A=1$ and $B=0$ yields

$$x_{n+1}=\frac{1}{x_n} \quad \text{and} \quad x_n \neq 0 \quad \text{for all } n \geq 1.$$

Hence,

$$x_{n+2}=x_n \quad \text{for all } n \geq 1.$$

The required recurrence relation which generates a periodic sequence of period 2 is

$$x_n x_{n+1}=1, \quad \text{where} \quad x_1 \neq 0.$$

Next, from **(ii)**, as $A=-B^2$, then

$$f^3(x_n)=x_n, \quad \text{where} \quad x_n \neq \frac{1}{B}.$$

Setting $A=-1$ and $B=1$ yields the recurrence relation

$$-x_n x_{n+1}+x_{n+1}=1, \quad \text{where } x_1 \neq 1.$$

Hence,

$$\begin{aligned}
x_{n+1} &= \frac{1}{1-x_n} \\
&= \frac{1}{1-\frac{1}{1-x_{n-1}}} \\
&= 1-\frac{1}{x_{n-1}} \\
&= 1-\frac{1}{\frac{1}{1-x_{n-2}}} \\
&= x_{n-2}
\end{aligned}$$

which shows that the recurrence relation generates a periodic sequence of period 3.

Question 3

(i) Let Q_n denote the proposition

$$\int_0^t x^n e^{-x}\,dx = n!\left(1-e^{-t}P_n(t)\right)$$

for all non-negative integers n.

When $n=0$, we have

$$\text{LHS} = \int_0^t e^{-x}\,dx = 1-e^{-t} = \text{RHS}$$

so Q_0 is true.
Assume that Q_k is true for some non-negative integer k. That is,

$$\int_0^t x^k e^{-x}\,dx = k!\left(1-e^{-t}P_k(t)\right).$$

To prove that Q_{k+1} is true, we need to show

$$\int_0^t x^{k+1} e^{-x}\,dx = (k+1)!\left(1-e^{-t}P_{k+1}(t)\right).$$

Consider Q_{k+1}. Then,

$$\begin{aligned}
\text{LHS} &= \int_0^t x^{k+1}e^{-x}\,dx \\
&= \left[x^{k+1}\left(-e^{-x}\right)\right]_0^t + (k+1)\int_0^t x^k e^{-x}\,dx \\
&= -t^{k+1}e^{-t} + (k+1)\left[k!\left(1-e^{-t}\mathrm{P}_k(t)\right)\right] \quad \text{by induction hypothesis} \\
&= -t^{k+1}e^{-t} + (k+1)! - e^{-t}(k+1)\sum_{i=0}^{k}\frac{t^i}{i!} \\
&= (k+1)! - e^{-t}\left[t^{k+1} + (k+1)\sum_{i=0}^{k}\frac{t^i}{i!}\right] \\
&= (k+1)! - e^{-t}\left[(k+1)!\sum_{i=0}^{k+1}\frac{t^i}{i!}\right] \\
&= (k+1)!\left(1-e^{-t}P_{k+1}(t)\right)
\end{aligned}$$

Since Q_0 is true and Q_k is true implies Q_{k+1} is true, then by induction, Q_n is true for all non-negative integers n.

(ii) Note that $\sum_{i=0}^{n}\frac{t^i}{i!}$ is the partial sum of the Maclaurin series of e^t so

$$\sum_{i=0}^{n}\frac{t^i}{i!} < e^t.$$

Hence,

$$\begin{aligned}
\int_0^\infty x^n e^{-x}\,dx &= \lim_{t\to\infty}\left[n!\left(1-e^{-t}P_n(t)\right)\right] \\
&= n! - \lim_{t\to\infty} e^{-t}\sum_{i=0}^{n}\frac{t^i}{i!} \\
&= n!
\end{aligned}$$

(iii) By the binomial theorem,

$$\begin{aligned}
\left(1+\frac{t}{n}\right)^n &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{t}{n}\right)^i \\
&= \sum_{i=0}^{n} \frac{n\,(n-1)\,(n-2)\ldots(n-i+1)}{i!} \left(\frac{t}{n}\right)^i \\
&= \sum_{i=0}^{n} (1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{i-1}{n}\right)\left(\frac{t}{n}\right)^i \\
&\leq \sum_{i=0}^{n} \left(\frac{t}{n}\right)^i \\
&\leq \sum_{i=0}^{n} \frac{t^i}{i!} \\
&= P_n(t)
\end{aligned}$$

Now, we ascertain the upper bound for $P_n(t)$.

$$\begin{aligned}
\left(1-\frac{t}{n}\right)^{-n} &= 1+(-n)\left(-\frac{t}{n}\right)+\frac{(-n)\,(-n-1)}{2!}\left(-\frac{t}{n}\right)^2+\frac{(-n)\,(-n-1)\,(-n-2)}{3!}\left(-\frac{t}{n}\right)^3+\ldots \\
&= 1+t+\frac{n\,(n+1)}{2!}\left(\frac{t}{n}\right)^2+\frac{n\,(n+1)\,(n+2)}{3!}\left(\frac{t}{n}\right)^3+\ldots \\
&= 1+t+\frac{t^2}{2!}\left(1+\frac{1}{n}\right)+\frac{t^3}{3!}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\ldots \\
&> 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots \quad \text{since } 1+\frac{k}{n}>1 \text{ for all } k\in\mathbb{Z}^+ \\
&= e^t \\
&> \sum_{i=0}^{n} \frac{t^i}{i!} \\
&= \mathrm{P}_n(t)
\end{aligned}$$

Question 4

(i) We have $23y = 13(144-3x-5z)$, which implies $23y$ is a multiple of 13. However, as $\gcd(23,13)=1$, then y is a multiple of 13. Since y is prime, then $y=13$.

(ii) **(a)** We have $3x+5z=121$. In modulo 5, this equation becomes $3x \equiv 1 \pmod 5$. Consider the following table:

$x \pmod 5$	$3x \pmod 5$
0	0
1	3
2	1
3	4
4	2

It follows that $x \equiv 2 \pmod 5$.

In modulo 3, $3x+5z=121$ can be written as $2z \equiv 1 \pmod 3$. Consider the following table:

$z \pmod 3$	$2z \pmod 3$
0	0
1	2
2	1

It follows that $z \equiv 2 \pmod 3$.

(b) From **(a)**, there exists $s,t \in \mathbb{Z}$ such that $x=5s+2$ and $z=3t+2$.
Since $3x+5z=121$, then $3(5s+2)+5(3t+2)=121$. As such, $s+t=7$.
So, $|z-x| = |3t-5s| = |3(7-s)-5s| = |21-8s|$.
The minimum value of $|z-x|$ is obtained when $s=3$, so $x=17$. Consequently, $z=14$. Therefore, $(x,y,z)=(17,13,14)$.

(iii) From **(i)**, since y is prime, then $y=13$. We now find solutions to $3x+5z=121$ such that x and z are prime.
As $x=\frac{121-5z}{3}$, then $x \leq 40\frac{1}{3}$. We consider the primes of the form 2 modulo 5 and are less than 40, which are 2, 7, 17, and 37.
If $x=2$, then $z=23$ which is prime. If $x=7$, then $z=20$ which is not prime. If $x=17$, then $z=14$ which is not prime. Lastly, if $x=37$, then $z=2$ which is prime.
We conclude that $(x,y,z)=(2,13,23),(37,13,2)$ are the only solutions.

Question 5

(a) **(i)** Replace x with $\frac{1}{x}$ and the result follows.

(ii) We have

$$f(x)+2f\left(\frac{1}{x}\right)=3x \quad 1$$

$$f\left(\frac{1}{x}\right)+2f(x)=\frac{3}{x} \quad 2$$

2×2 yields $2f\left(\frac{1}{x}\right)+4f(x)=\frac{6}{x}$. So, $3f(x)=\frac{6}{x}-3x$. It follows that $f(x)=\frac{2}{x}-x$.

(b) As mentioned in the question,

$$g(x)+g(-x)+g\left(\frac{1}{x}\right)=x \quad 3.$$

Replacing x with $-x$ in 3 yields

$$g(-x)+g(x)+g\left(-\frac{1}{x}\right)=-x \quad 4.$$

Adding 3 and 4 yields

$$g\left(\frac{1}{x}\right)+g\left(-\frac{1}{x}\right)=-2g(x)-2g(-x) \quad 5.$$

Replacing x with $\frac{1}{x}$ in 3 yields

$$\begin{aligned} g\left(\frac{1}{x}\right)+g\left(-\frac{1}{x}\right)+g(x) &= \frac{1}{x} \\ -g(x)-2g(-x) &= \frac{1}{x} \quad \text{by 5} \end{aligned}$$

Denote $-g(x)-2g(-x)=\frac{1}{x}$ by 6. Replacing x with $-x$ in 6 yields $-g(-x)-2g(x)=-\frac{1}{x}$. Let this equation be 7. So, $7-2\times 6$ yields $3g(-x)=-\frac{3}{x}$. It follows that $g(x)=\frac{1}{x}$.

Question 6

(i) Note that

$$\begin{aligned} x_{n+1}^2 - x_n x_{n+2} &= x_{n+1}^2 - x_n\left(dx_{n+1}-x_n\right) \\ &= x_{n+1}^2 + x_n^2 - dx_n x_{n+1} \end{aligned}$$

Now, we prove by induction that $x_{n+1}^2 - x_n x_{n+2} = D$ for all positive integers n. Let P_n denote this proposition.
P_1 is true as $x_2^2 - x_1 x_3 = D$ as mentioned in the question.
Assume P_k is true for some positive integer k. That is, $x_{k+1}^2 - x_k x_{k+2} = D$.
To show P_{k+1} is true, we need to show $x_{k+2}^2 - x_{k+1}x_{k+3} = D$.

$$\begin{aligned} x_{k+2}^2 - x_{k+1}x_{k+3} &= x_{k+2}^2 - x_{k+1}\left(dx_{k+2}-x_{k+1}\right) \quad \text{by definition of recurrence relation} \\ &= x_{k+2}^2 + x_{k+1}^2 - dx_{k+1}x_{k+2} \\ &= x_{k+2}^2 + x_{k+1}^2 - x_k x_{k+2} + x_k x_{k+2} - dx_{k+1}x_{k+2} \\ &= x_{k+2}^2 + D + x_k x_{k+2} - dx_{k+1}x_{k+2} \quad \text{by induction hypothesis} \\ &= D + x_{k+2}^2 + x_k x_{k+2} - dx_{k+1}x_{k+2} \\ &= D + x_{k+2}\left(x_{k+2} + x_k - dx_{k+1}\right) \\ &= D \end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n.

(ii) Set $x_n = 0$ and we obtain $x_n x_{n+2} = 0$. Hence, $D = x_{n+1}^2$, which is a perfect square.

(iii)
- **Case 1:** Suppose the sequence contains a zero term. By **(ii)**, D is a perfect square.
- **Case 2:** Suppose the sequence does not contain any zero terms. So, it contains both positive and negative terms. Then, there exists a positive integer n such that x_n and x_{n+1} have different signs.

  To justify this, we prove by contradiction. Suppose on the contrary that $x_1, x_2, x_3, \ldots$ all have the same sign. Then, the sequence contains either only positive or negative terms, which is a contradiction. As such, $x_n x_{n+1} \leq -1$, implying that $-dx_n x_{n+1} \geq d$ since $d > 0$.

Since $x_n, x_{n+1} \in \mathbb{Z}$, then the sum of their squares is at least 2. Hence,

$$\begin{aligned} D &= x_n^2 + x_{n+1}^2 - dx_n x_{n+1} \\ &\geq 2 + d \end{aligned}$$

(iv) As $x_n x_{n+1} \leq -1$, set $x_n = 1$ and $x_{n+1} = 1$.
It is easy to show that the five successive terms, by substitution, are $1, -1, -4, -11, -29$.

Question 7

(i) Take some element in X. It can be mapped to Y via n ways. Repeat this for the remaining $m-1$ elements in X.
It follows that the number of functions that map X to Y is n^m.

(ii) Take some element in X, which can be mapped to Y via n ways.
Take another element in X, which can be mapped to one of the remaining $n-1$ elements in Y.
Repeating this process, the last element in X can be mapped to either of the remaining $n-m+1$ elements in Y.
It follows that the number of one-to-one functions from X to Y is $n(n-1)(n-2)\ldots(n-m+1) = \frac{n!}{(n-m)!}$.

(iii) Let A_i be the event that $y_i \in Y$ does not get mapped from any element in X, where $1 \leq i \leq n$. Note that $\{y_1, \ldots, y_n\}$ is a permutation of $\{1, \ldots, n\}$.

We wish to find $|A_1' \cap \ldots \cap A_n'|$, for which by de Morgan's law, is

$$n(S) - \left| \bigcup_{i=1}^{n} A_i \right|.$$

From **(i)**, $n(S) = n^m$.
Also,

$$\begin{aligned} \sum_{i=1}^{n} |A_i| &= \binom{n}{1}(n-1)^m \\ \sum_{i<j} |A_i \cap A_j| &= \binom{n}{2}(n-2)^m \\ \sum_{i<j<k} |A_i \cap A_j \cap A_k| &= \binom{n}{3}(n-3)^m \end{aligned}$$

By the principle of inclusion and exclusion,

$$\begin{aligned} n(S) - \left|\bigcup_{i=1}^{n} A_i\right| &= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \ldots + (-1)^{n-1}\binom{n}{n-1}1^m \\ &= \sum_{r=0}^{n-1}(-1)^k\binom{n}{r}(n-r)^m \end{aligned}$$

(iv) Since $m = n = 5$, the number of one-to-one functions is $5!$.
First, we subtract all functions where each element is mapped to itself, for which there are $\binom{5}{1}(5-1)!$ of them.
Then, add all functions consisting of two elements that are mapped to themselves due to overcounting previously. We thus add $\binom{5}{2}(5-2)!$.

It follows by the principle of inclusion and exclusion that the required number of one-to-one functions mapping X to Y which map no element to itself is

$$5! - \binom{5}{1}(5-1)! + \binom{5}{2}(5-2)! - \binom{5}{3}(5-3)! + \binom{5}{4}(5-4)! - \binom{5}{5}(5-5)! = 44.$$

Remark for Question 7: For **(iv)**, this can be also thought of as the number of derangements of a set with 5 elements. It is a known result that the number of derangements of an n-element set is given by $\sum_{r=0}^{n}\frac{(-1)^r}{r!}$ which follows by the principle of inclusion and exclusion. Substituting $n = 5$, the result follows.

Question 8

(a) An ellipse has two lines of symmetry which are along its major axis and along its minor axis. So, if we rotate the point with position vector $\mathbf{x}$ (which lies in F) about the origin by 180°, we obtain the point with position vector $-\mathbf{x}$ which also lies in F.

As

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\mathbf{y}-\mathbf{x}}{2} = \frac{\mathbf{y}+(-\mathbf{x})}{2},$$

it implies that the point with position vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is the midpoint of the points with position vectors $-\mathbf{x}$ and $\mathbf{y}$.

In fact, the line segment connecting the points with position vectors $-\mathbf{x}$ and $\mathbf{y}$ lies entirely in F as F is convex.

(b) **(i)** We first prove that any coordinate on E which undergoes a transformation can lie on any lattice point contained within the 2×2 square centred on the origin.

Define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2\lfloor x/2 \rfloor \\ 2\lfloor y/2 \rfloor \end{pmatrix}.$$

Suppose on the contrary that the ellipse has an area larger than 4. Then, there exist some lattice points other than the origin contained in the 2×2 square.

As $\left(2\left\lfloor \frac{x_1}{2} \right\rfloor, 2\left\lfloor \frac{y_1}{2} \right\rfloor\right)$ is a lattice point on the 2×2 square, then $\left(-2\left\lfloor \frac{x_2}{2} \right\rfloor, -2\left\lfloor \frac{y_2}{2} \right\rfloor\right)$ is also a lattice point. Suppose they have position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.

Using **(a)**, we establish that the point with position vector $\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ must also lie in E. That is,

$$\left(\left\lfloor \frac{x_1}{2} \right\rfloor - \left\lfloor \frac{x_2}{2} \right\rfloor, \left\lfloor \frac{y_1}{2} \right\rfloor - \left\lfloor \frac{y_2}{2} \right\rfloor\right)$$

is a lattice point in the ellipse of area larger than 4.

(ii) Note that C has an area of $4p$ units^2. Since $p \in \mathbb{Z}^+$, then the area of C must be at least 4 units^2. Consider the position vector of the required coordinate. That is,

$$\begin{pmatrix} mp - nu \\ n \end{pmatrix} = m\begin{pmatrix} p \\ 0 \end{pmatrix} - n\begin{pmatrix} u \\ 1 \end{pmatrix}.$$

This changes the basis from the standard basis vectors

$$\mathbf{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to

$$\begin{pmatrix} p \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u \\ 1 \end{pmatrix}$$

respectively.

So, the vector space $\mathbb{R}^2$ is now tiled by parallelograms instead of unit squares. Since every lattice point must be a vertex of one of these parallelograms, we conclude that a parallelogram lies completely inside C and the result follows.

(c) Since u^2+1 is an integer multiple of p, there exists $\lambda \in \mathbb{Z}$ such that $u^2+1=\lambda p$. Consider $x = mp - nu$ and $y = n$. Then,

$$\begin{aligned} x^2+y^2 &= m^2p^2 - 2mnpu + n^2u^2 + n^2 \\ &= m^2p^2 - 2mnpu + n^2\left(u^2+1\right) \\ &= m^2p^2 - 2mnpu + n^2\lambda p \\ &= p\left(m^2p + n^2\lambda - 2mnu\right) \\ &\equiv 0 \pmod{p} \end{aligned}$$

By considering the radius of the circle,

$$x^2+y^2 < \left(2\sqrt{\frac{p}{\pi}}\right)^2 = \frac{4p}{\pi}.$$

As $x^2+y^2>0$, then $0<x^2+y^2<\dfrac{4p}{\pi}$.
Lastly, since $p<\dfrac{4p}{\pi}<2p$, we have $x^2+y^2=p$.

Remark for Question 8: For **(a)**, I created an interactive simulation on Desmos. Here, we consider the general Cartesian form of a conic section which is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

and all the coefficients are real and A,B,C are all non-zero. Since the ellipse is centred on the origin, then $C=D=0$.

An ellipse is an example of a conic section. Since the conic is non-degenerate, we have

$$ACF+\frac{1}{4}\left(BDE-AE^2-B^2F-CD^2\right)\neq 0.$$

Also, since the conic is an ellipse, we have $4AC-B^2>0$.

I found a post on StackExchange which is related to **(bii)** and **(c)**. This question has some semblance to Minkowski's theorem. The convex body theorem for lattices in $\mathbb{R}^2$ is as follows. Suppose L is a lattice in $\mathbb{R}^2$ defined as $L=\{m\mathbf{v}_1+n\mathbf{v}_2 : m,n\in\mathbb{Z}\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors. That is, we cannot express $\mathbf{v}_1$ as a scalar multiple of $\mathbf{v}_2$ and vice versa. Let d be the area of a fundamental parallelogram of L. If S is a convex and origin-symmetric region with $\text{Area}(S) > 4d$, then S contains some point q, other than the origin, such that $q \in L$.

The reader can check out Blichfeldt's Theorem too.

Chapter 7

2021 Paper Solutions

Question 1

(a) **(i)** Suppose $\dfrac{\ln x}{1+x^2} = 0$. Then, $\ln x = 0$, so $x = 1$.

The area of R is $-\displaystyle\int_0^1 \frac{\ln x}{1+x^2}\,dx$, whereas the area of S is $\displaystyle\int_1^\infty \frac{\ln x}{1+x^2}\,dx$.

By considering the area of S, letting $x = \frac{1}{t}$, we have $\frac{dx}{dt} = -\frac{1}{t^2}$. So,

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{1+x^2}\,dx &= \int_1^0 \frac{\ln\left(\frac{1}{t}\right)}{1+\left(\frac{1}{t}\right)^2} \cdot \left(-\frac{1}{t^2}\right)\,dt \\
&= \int_0^1 \frac{\ln 1 - \ln t}{1+t^2}\,dt \\
&= -\int_0^1 \frac{\ln x}{1+x^2}\,dx
\end{aligned}$$

(ii) Using the substitution $x = at$, we have $dx = a\,dt$.

The integral becomes

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{a^2+x^2}\,dx &= \int_0^\infty \frac{\ln(at)}{a^2+a^2t^2} \cdot a\,dt \\
&= \frac{1}{a}\int_0^\infty \frac{\ln a + \ln t}{1+t^2}\,dt \\
&= \frac{1}{a}\left(\ln a \int_0^\infty \frac{1}{1+t^2}\,dt + \int_0^\infty \frac{\ln t}{1+t^2}\,dt\right) \\
&= \frac{\ln a}{a}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi \ln a}{2a}
\end{aligned}$$

(b)

$$\begin{aligned}\int_0^\infty \ln\left(\frac{a^2+x^2}{x^2}\right)\,dx &= \left[x\ln\left(\frac{a^2+x^2}{x^2}\right)\right]_0^\infty + 2a^2\int_0^\infty \frac{1}{a^2+x^2}\,dx\\ &= 2a^2\int_0^\infty \frac{1}{a^2+x^2}\,dx\\ &= 2a\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty\\ &= a\pi\end{aligned}$$

Question 2

(a) Without loss of generality, let $a \geq b \geq c > 0$. Then,

$$a^r(a-b)(a-c)+b^r(b-c)(b-a)+c^r(c-a)(c-b) = (a-b)[a^r(a-c)-b^r(b-c)]+c^r(c-a)(c-$$

Note that $c-a \leq 0$ and $c-b \leq 0$ so $c^r(c-a)(c-b) \geq 0$. It suffices to show that

$$a^r(a-c)-b^r(b-c) \geq 0.$$

In other words,

$$\left(\frac{a}{b}\right)^r \geq 1 \geq \frac{b-c}{a-c}.$$

$\left(\frac{a}{b}\right)^r \geq 1$ is merely a consequence of $a \geq b$ and $r > 0$. To see why $1 \geq \frac{b-c}{a-c}$, we see that the inequality is equivalent to $a-c \geq b-c$, which is true because $a \geq b$.

(b) **(i)** Note that $3abc$ can be written as $abc+abc+abc$. Suppose $a^3+abc-a^2(b+c) = a^r(a-b)(a-c)$. Then, $a(a^2+bc-ab-ac) = a^r(a-b)(a-c)$. Observe that $a^2+bc-ab-ac$ factorises as $(a-b)(a-c)$, so we can set $r=1$.

Thus, the inequality follows by setting $r=1$ in **(a)**. In particular,

$$\begin{aligned}a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) &\geq 0\\ a(a^2-ab-ac+bc)+b(b^2-ab-bc+ac)+c(c^2-ac-bc+ab) &\geq 0\end{aligned}$$

so it follows that

$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b).$$

(ii) Consider $\dfrac{a+b+c}{a^2b^2c^2}$. Note that $\dfrac{a}{a^2b^2c^2} - \dfrac{b^2+c^2}{a^3b^2c^2} = \dfrac{a^2-b^2-c^2}{a^3b^2c^2}$. So,

$$\begin{aligned}\frac{1}{a^5}+\frac{a}{a^2b^2c^2}-\frac{b^2+c^2}{a^3b^2c^2} &= \frac{1}{a^3}\left(\frac{1}{a^2}+\frac{a^2-b^2-c^2}{b^2c^2}\right)\\ &= \frac{(a+c)(a-c)(a+b)(a-b)}{a^5b^2c^2}\end{aligned}$$

Note that $a+c, a+b, a^5b^2c^2 \geq 0$, so it suffices to prove that

$$(a-b)(a-c)+(b-c)(b-a)+(c-a)(c-b) \geq 0.$$

This is clear by setting $r=1$ in **(a)**.

Remark for Question 2: The inequality in **(a)** is known as Schur's inequality.

Question 3

(a) Let P_n be the proposition that

$$v\frac{d^{n+2}y}{dx^{n+2}}+(n+2)\frac{dv}{dx}\frac{d^{n+1}y}{dx^{n+1}}+\binom{n+2}{2}\frac{d^2v}{dx^2}\frac{d^ny}{dx^n}=0$$

for all positive integers n, given that $y=\frac{u}{v}$.

We use the notation $u'=\frac{du}{dx}$, as well as $v'=\frac{dv}{dx}$. Note that $u=vy$. When $n=1$, we have

$$\begin{aligned}
u' &= vy'+v'y\\
u'' &= vy''+2v'y'+v''y\\
u''' &= vy'''+v'y''+2v'y''+2v''y'+v'''y+v''y'\\
&= vy'''+3v'y''+3v''y' \text{ since } v \text{ is quadratic} \implies v'''=0
\end{aligned}$$

Since u is quadratic, then $u'''=0$, so it follows that

$$v\frac{d^3y}{dx^3}+3\frac{dv}{dx}\frac{d^2y}{dx^2}+3\frac{d^2v}{dx^2}\frac{dy}{dx}=0.$$

As such, P_1 is true.

Suppose P_k is true. That is to say,

$$v\frac{d^{k+2}y}{dx^{k+2}}+(k+2)\frac{dv}{dx}\frac{d^{k+1}y}{dx^{k+1}}+\binom{k+2}{2}\frac{d^2v}{dx^2}\frac{d^ky}{dx^k}=0.$$

To prove P_{k+1} is true, we need to show that

$$v\frac{d^{k+3}y}{dx^{k+3}}+(k+3)\frac{dv}{dx}\frac{d^{k+2}y}{dx^{k+2}}+\binom{k+3}{2}\frac{d^2v}{dx^2}\frac{d^{k+1}y}{dx^{k+1}}=0.$$

From P_k, we first differentiate both sides to obtain

$$vy^{(k+3)}+v'y^{(k+2)}+(k+2)v'y^{(k+2)}+(k+2)v''y^{(k+1)}+\binom{k+2}{2}v''y^{k+1}+\binom{k+2}{2}v'''y^{(k+1)}=0.$$

Since $v'''=0$, then

$$vy^{(k+3)}+(k+3)v'y^{(k+2)}+\left[k+2+\binom{k+2}{2}\right]v''y^{(k+1)}=0.$$

Observe that

$$\begin{aligned}
k+2+\binom{k+2}{2} &= k+2\left(\frac{2+k+1}{2}\right)\\
&= \frac{(k+3)(k+2)}{2}\\
&= \binom{k+3}{2}
\end{aligned}$$

and the result follows.

(b) We first prove that z_n is an arithmetic progression.
Since $v = (\alpha - x)^2$, then $\frac{dv}{dx} = 2x - 2\alpha$, so $\frac{d^2v}{dx^2} = 2$. From **(a)**,

$$(\alpha - x)^2 \frac{d^{n+2}y}{dx^{n+2}} + 2(n+2)(x-\alpha)\frac{d^{n+1}y}{dx^{n+1}} + (n+2)(n+1)\frac{d^n y}{dx^n} = 0.$$

So,

$$\begin{aligned}(\alpha - x)^2 \frac{(n+2)!z_{n+2}}{(\alpha - x)^{n+4}} + 2(x-\alpha)\frac{(n+2)!z_{n+1}}{(\alpha - x)^{n+3}} + \frac{(n+2)!z_n}{(\alpha - x)^{n+2}} &= 0 \\ (n+2)!z_{n+2} - 2(n+2)!z_{n+1} + (n+2)!z_n &= 0 \\ z_{n+2} - z_{n+1} &= z_n - z_{n+1}\end{aligned}$$

It follows that the difference of consecutive terms is a constant.

Now, write

$$y = \frac{u}{(\alpha - x)^2} = \frac{A}{\alpha - x} + \frac{B}{(\alpha - x)^2}.$$

Then,

$$\frac{dy}{dx} = \frac{A}{(\alpha - x)^2} + \frac{2B}{(\alpha - x)^3} \text{ and } \frac{d^2y}{dx^2} = \frac{2A}{(\alpha - x)^3} + \frac{6B}{(\alpha - x)^4}.$$

As such,

$$\begin{aligned} z_2 - z_1 &= \frac{(\alpha - x)^4}{2}\frac{d^2y}{dx^2} - (\alpha - x)^3\frac{dy}{dx} \\ &= \frac{(\alpha - x)^4}{2}\left[\frac{2A}{(\alpha - x)^3} + \frac{6B}{(\alpha - x)^4}\right] - (\alpha - x)^3\left[\frac{A}{(\alpha - x)^2} + \frac{2B}{(\alpha - x)^3}\right] \\ &= A(\alpha - x) + 3B - A(\alpha - x) - 2B \\ &= B \end{aligned}$$

Recall that $u = A(\alpha - x)^2 + B$. Setting $u = B$, we have $x = \alpha$ so it follows that the common difference is $u(\alpha)$.

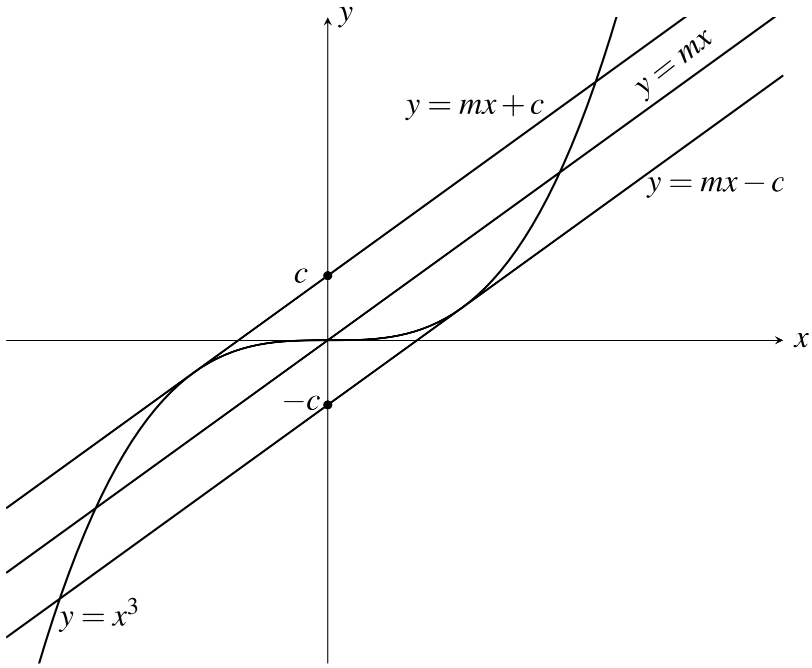
Question 4

(a) Given $y = x^3$, we have $\frac{dy}{dx} = 3x^2$.
Note that the curve passes through $\left(x_0, x_0^3\right)$, where x_0 is arbitrary. So, $m = 3x_0^2$.
The equation of the tangent is $y - x_0^3 = 3x_0^2(x - x_0)$. Comparing this with $y = mx + c$, we have $m = 3x_0^2$ and $c = -2x_0^3$.
As such,

$$\begin{aligned}\left(\frac{m}{3}\right)^3 &= \left(-\frac{c}{2}\right)^2 \\ \frac{m^3}{27} &= \frac{c^2}{4}\end{aligned}$$

The result follows.

(b) Consider the following sketch:



$y = mx + c$ and $y = x^3$ intersect at three points. So, $|c| < 2x_0^3$.
Squaring both sides, then multiplying by 27 yields $27c^2 < 27\left(2x_0^3\right)^2 = 108x_0^6$.
From **(a)**, since $m = 3x_0^2$, then $108x_0^6 = 108\,(m/3)^3 = 4m^3$. It follows that $27c^2 < 4m^3$.

(c) The standard equation of a circle centred at (a,b) with radius r is $(x-a)^2 + (y-b)^2 = r^2$.
Since the circle passes through the origin, then $a^2 + b^2 = r^2$.

We consider the parabola $y = x^2$. Substituting this into the equation of the circle, we have

$$\begin{aligned} (x-a)^2 + \left(x^2 - b\right)^2 &= r^2 \\ x^2 - 2ax + a^2 + x^4 - 2bx^2 + b^2 - r^2 &= 0 \\ x^4 + (1-2b)\,x^2 - 2ax &= 0 \quad \text{since } a^2 + b^2 = r^2 \end{aligned}$$

So either $x = 0$ or $x^3 + (1-2b)\,x - 2a = 0$. From **(b)**, $x^3 = (2b-1)\,x + 2a$ has three distinct roots if $27\,(2a)^2 < 4\,(2b-1)^3$, so the coordinates of the centre of the circle (a,b) satisfy inequality

$$27a^2 < (2b-1)^3\,, \quad \text{where } a \neq 0.$$

The possible positions can be described by the following set:

$$\left\{(a,b) \in \mathbb{R}\backslash\{0\} \times \mathbb{R} : b > \frac{1}{2}\left(3a^{2/3} + 1\right)\right\}$$

Question 5

(a) Note that $\gcd(a+b, c+d) = z$. So, z divides $a+b$ and z also divides $c+d$. Since z also divides any linear combination of $a+b$ and $c+d$, by observing that

$$-c(a+b)+a(c+d) = ad-bc,$$

we infer that z also divides $ad-bc$. As such, there exist $\lambda, \mu \in \mathbb{N}$ such that $a+b = \lambda z$ and $c+d = \mu z$, where $\gcd(\lambda, \mu) = 1$. Note that if $\gcd(\lambda, \mu) > 1$, it would contradict the fact that $z = \gcd(a+b, c+d)$. We write $ad-bc = wz$ for some $w \in \mathbb{N}$. It follows that $w^2 = \lambda\mu$.

We consider two cases.

- **Case 1:** Suppose w divides λ. Then, $\lambda = w^2$ and $\mu = 1$. So, $a+b = w^2 z$ and $c+d = z$. Setting $x = w$ and $y = 1$, the result follows. If w divides μ instead, we can argue similarly and the result follows.
- **Case 2:** Suppose w does not divide λ and μ. Then λ and μ must be perfect squares. So, there exist $x, y \in \mathbb{N}$ such that $\lambda = x^2$ and $\mu = y^2$. The result follows.

(b) Consider

$$\alpha\left(\frac{y}{x}\right)^2 + \beta\left(\frac{y}{x}\right) + \gamma = 0,$$

where α, β, γ are constants and $\alpha \neq 0$. Multiplying both sides by x^2, we have $\alpha y^2 + \beta xy + \gamma x^2 = 0$. From **(a)**, since

$$x = \sqrt{\frac{a+b}{z}} \text{ and } y = \sqrt{\frac{c+d}{z}},$$

we have

$$\alpha\left(\frac{c+d}{z}\right) + \beta\sqrt{\left(\frac{a+b}{z}\right)\left(\frac{c+d}{z}\right)} + \gamma\left(\frac{a+b}{z}\right) = 0.$$

So, $\alpha(c+d) + \gamma(a+b) + \beta(ad-bc) = 0$. As mentioned at the start of **(a)**, we can set $\alpha = a$, $\beta = -1$ and $\gamma = -c$. So, the required quadratic equation is

$$a\left(\frac{y}{x}\right)^2 - \frac{y}{x} - c = 0.$$

As such,

$$\frac{y}{x} = \frac{1 \pm \sqrt{4ac+1}}{2a}.$$

Since $x, y \in \mathbb{N}$, then $\frac{y}{x}$ is rational. So,

$$4ac+1 = \left(\frac{2ay}{x} - 1\right)^2,$$

where $\frac{2ay}{x} - 1$ is rational, so $4ac+1$ is a perfect square.

Question 6

(a) Consider $2 \times 3 \times 5 = 30$. There are 5 ways to write express $2 \times 3 \times 5$ as a product of 3 positive integers where the order of these integers does not matter as seen below.

$$\begin{aligned} 30 &= 1 \times 1 \times 30 \\ &= 1 \times 2 \times 15 \\ &= 1 \times 3 \times 10 \\ &= 1 \times 6 \times 5 \\ &= 2 \times 3 \times 5 \end{aligned}$$

(b) To obtain $F(n)$, there are n cases to consider.

- **Case 1:** Suppose we have a product of $n-1$ distinct primes, so the product is given by $p_1 p_2 \dots p_{n-1}$. Multiply this by some other prime p_n. There are

$$F(n-1) = \binom{n-1}{n-1} F(n-1)$$

ways to do this.

- **Case 2:** Choose some prime p_i, where $1 \leq i \leq n-1$, to be multiplied by p_n to obtain $p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_{n-1} (p_i p_n)$. There are

$$\binom{n-1}{1} F(n-2)$$

ways to do this.

- **Case 3:** Choose two distinct primes p_i, p_j, where $1 \leq i < j \leq n-1$ to be multiplied by p_n to obtain

$$p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_{j-1} p_{j+1} \cdots p_{n-1} (p_i p_j p_n).$$

There are

$$\binom{n-1}{2} F(n-3)$$

ways to do this.

Repeat this till the last case, where $p_1 p_2 \dots p_{n-1}$ is multiplied by p_n. This contributes to the $F(0)$ term in the sum.

Therefore,

$$\begin{aligned} F(n) &= \binom{n-1}{n-1} F(n-1) + \binom{n-1}{1} F(n-2) + \binom{n-3}{2} F(n-3) + \ldots + F(0) \\ &= \binom{n-1}{n-1} F(n-1) + \binom{n-1}{n-2} F(n-2) + \binom{n-1}{n-3} F(n-3) + \ldots + \binom{n-1}{0} F(0) \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} F(i) \end{aligned}$$

where we used the symmetry of binomial coefficients.

(c) Let $A = \alpha_1 \times \alpha_2 \times \ldots \times \alpha_{n-2}$, which is a product of $n-2$ positive integers. There is no duplication if the product is of the form $A \times p_{n-1} \times p_{n-1}$ (contributes to $F(n-2)$ ways) or $A \times p_{n-1}^2 \times 1$ (contributes to $F(n-1)$ ways). The result follows.

(d) **(i)** Note that $210 = 2 \times 3 \times 5 \times 7$ which factors into four distinct primes. Hence, the answer is $F(4) = 15$ (formula given in **(b)**.

(ii) We see that

$$\begin{aligned}
150 &= 1 \times 1 \times 1 \times 150 \\
&= 1 \times 1 \times 2 \times 75 \\
&= 1 \times 1 \times 5 \times 30 \\
&= 1 \times 1 \times 6 \times 25 \\
&= 1 \times 1 \times 10 \times 15 \\
&= 1 \times 2 \times 3 \times 25 \\
&= 1 \times 2 \times 5 \times 15 \\
&= 1 \times 3 \times 5 \times 10 \\
&= 1 \times 5 \times 5 \times 6 \\
&= 2 \times 3 \times 5 \times 5
\end{aligned}$$

so there are 10 ways.

Remark for Question 6: $F(n)$ can also be thought of as the n^{th} Bell number. The Bell numbers are used to count the number of partitions of a set.

Question 7

(a) Suppose we fix x. Then, xy_k is unique and there are $p-1$ possible products for a given x. Also, xy_k is not congruent to $0 \pmod p$ as $x, y_k \in Q$. Suppose on the contrary that none of the products xy_k is congruent to $1 \pmod p$. Then, each product is congruent to either

$$2 \pmod p \quad \text{or} \quad 3 \pmod p \quad \text{or} \quad \ldots \quad \text{or} \quad p-1 \pmod p.$$

There are $p-1$ possible products and $p-2$ numbers in $[2, p-1]$.
By the pigeonhole principle, there exists $y_i, y_j \in Q$ such that $xy_i \equiv xy_j \equiv k \pmod p$ for some $2 \leq k \leq p-1$. So, $xy_i = xy_j$, implying that $y_i = y_j$. Thus, there exists at least one $y \in Q$ such that $xy_i \equiv 1 \pmod p$.

In fact, y is unique. Suppose there exists $y_i, y_j \in Q$ such that $xy_i \equiv xy_j \equiv 1 \pmod p$. So, $x(y_i - y_j) \pmod p$. Either x is a multiple of p or $y_i - y_j$ is a multiple of p. Since $x \in Q$, then x cannot be a multiple of p, so it forces $y_i - y_j = 0$, implying that $y_i = y_j$. This establishes the uniqueness of y such that $xy \equiv 1 \pmod p$.

(b) There are $p-1$ choices for x and $p-1$ choices for y, so there are $(p-1)^2$ choices for xy.

- **Case 1:** Suppose $xy \in Q$. Then, by **(a)**, because $xyz = (xy)z$, it follows that there are $(p-1)^2$ choices for x, y, z such that $xyz \equiv 1 \pmod{p}$.
- **Case 2:** Suppose $xy \notin Q$. Then, we can always reduce the equation modulo p. That is, there exists $\lambda \in Q$ such that $xy - \lambda p \in Q$. From **(a)**, there exists precisely one $z \in Q$ such that $(xy - np)z \equiv 1 \pmod{p}$. Since $npz \equiv 0 \pmod{p}$, it follows that $xyz \equiv 1 \pmod{p}$.
- We consider three cases.
  - **Case 1:** Suppose x, y, z are all identical. Then, it reduces to finding all $x \in Q$ such that $x^3 \equiv 1 \pmod{p}$. Based on the preamble, it is clear that the number of choices is N.
  - **Case 2:** Suppose 2 of the x, y, z are identical. Then, we wish to find all $x, y \in Q$ such that $x^2 y \equiv 1 \pmod{p}$. The number of choices is $3(p-1-N)$.
  - **Case 3:** Suppose none of the x, y, z are identical. In other words, all three of them are distinct. We wish to find an expression for the number of ways, say W, such that $xyz \equiv 1 \pmod{p}$.

  From **(b)**, the number of choices of integers $x, y, z \in Q$ such that $xyz \equiv 1 \pmod{p}$ is $(p-1)^2$. By the principle of inclusion and exclusion,

$$\begin{aligned} W &= (p-1)^2 - 3(p-1-N) - N \\ &= (p-1)(p-4) + 2N \end{aligned}$$

- From **(c)**, the number of ways to choose distinct $x, y, z \in \mathbb{Q}$ such that $xyz \equiv 1 \pmod{p}$ is divisible by 3 due to symmetry. As such,

$$\begin{aligned} (p-1)(p-4) + 2N &\equiv 0 \pmod{3} \\ (p-1)(p-1) - N &\equiv 0 \pmod{3} \\ (p-1)^2 &\equiv N \pmod{3} \\ N &\equiv (p-1)^2 \pmod{3} \quad \text{by symmetry of congruence} \end{aligned}$$

- From **(d)**, $N \equiv 0 \pmod{3}$, so N is a multiple of 3. What is more important is that $N \geq 3$. So, there exists at least three distinct $x \in Q$ such that $x^3 \equiv 1 \pmod{p}$. Choose $x \in Q$, where $x \neq 1$, such that $x^3 \equiv 1 \pmod{p}$. So, $x^3 - 1 \equiv 0 \pmod{p}$. By the difference of cubes formula, $(x-1)\left(x^2+x+1\right) \equiv 0 \pmod{p}$. So, p divides $(x-1)\left(x^2+x+1\right)$. By Euclid's lemma, p divides $x-1$ or p divides x^2+x+1. But, we have chosen x such that $x-1 \neq 0$. Since p is prime, we have p divides x^2+x+1 and the result follows.

Question 8

(a) Without loss of generality, suppose $e = e_1$. By the triangle inequality, $(e_1, e_2, \dots, e_m) \in P$ if and only if $e_2 + \dots + e_m > e$. Adding $e_1 = e$ to both sides, the result follows.

(b) For each Q_i, set $e_i = e$, so all the Q_i's are disjoint.
As the total number of m-tuples is N^m, the result follows.

(c) We consider three cases.

- **Case 1:** Suppose $1 \le i \le m-1$. Since $e_i \ge 1$ (the preamble states that $e_i \in \mathbb{Z}$ and it denotes length), then $1 + x_i \ge 1$, so $x_i \ge 0$.
- **Case 2:** Suppose $i = m$. Then,

$$x_m = e_m - e_1 - e_2 - \dots - e_{m-1}$$

  Since $(e_1, e_2, \dots, e_m) \in Q_m$, then $e_1 + e_2 + \dots + e_{m-1} < e_m$, where we chose $e = e_m$. The result follows.
- **Case 3:** For x_{m+1}, from **(b)**, we deduced that $N \ge e_m$, so $x_{m+1} \ge 0$.

(d)

$$\begin{aligned}
\sum_{i=1}^{m+1} x_i &= x_m + x_{m+1} + \sum_{i=1}^{m-1} x_i \\
&= e_m - \sum_{i=1}^{m-1} e_i + N - e_m + \sum_{i=1}^{m-1} (e_i - 1) \\
&= e_m - e_m + N + \sum_{i=1}^{m-1} (-e_i + e_i - 1) \\
&= N - m + 1
\end{aligned}$$

Next, we deduce the formula for $|Q_m|$. Consider the equation

$$x_1 + x_2 + \dots + x_{m+1} = N - m + 1.$$

The number of non-negative solutions $(x_1, \dots, x_{m+1})$ is the number of ways to distribute $N - m + 1$ identical balls into $m + 1$ distinct boxes, thus establishing a bijection.
As such,

$$|Q_m| = \binom{N+1}{m}.$$

(e) The total number of triangles that can be formed, including degenerate ones, is $10^3 = 1000$.
Setting $N = 10$, the number of 3-tuples that satisfy the triangle inequality is

$$\begin{aligned}
10^3 - |Q_1| - |Q_2| - |Q_3| &= 10^3 - 3\binom{11}{3} \quad \text{by symmetry as we can choose either } e_1, e_2, e_3 \text{ to be } e \\
&= 505
\end{aligned}$$

Chapter 8

2022 Paper Solutions

Question 1

(a) **(i)** Number of ways is $4^{10} = 1048576$

(ii) Let E_1, E_2, E_3, E_4 denote the following events:

E_1 denotes the event when no one obtained the A grade

E_2 denotes the event when no one obtained the B grade

E_3 denotes the event when no one obtained the C grade

E_4 denotes the event when no one obtained the D grade

We wish to find

$$\left|\bigcap_{i=1}^{4} E_i'\right| = 4^{10} - \left|\bigcup_{i=1}^{4} E_i\right| \quad \text{by de Morgan's law,}$$

so by the principle of inclusion and exclusion, we have

$$\begin{aligned}\left|\bigcup_{i=1}^{4} E_i\right| &= \sum_{i=1}^{4} |E_i| - \sum_{1\le i<j\le 4} \left|E_i \cap E_j\right| + \sum_{1\le i<j<k\le 4} \left|E_i \cap E_j \cap E_k\right| - \left|\bigcap_{i=1}^{4} E_i\right| \\ &= \binom{4}{1}3^{10} - \binom{4}{2}2^{10} + \binom{4}{3}1^{10} - 0 \\ &= 230056\end{aligned}$$

Hence, the answer is $4^{10} - 230056 = 818520$.

(iii) The Stirling numbers of the second kind account for the distribution of distinct objects into identical boxes.
So we divide **(ii)**'s answer by 4!, so $S(10,4) = 230056/4! = 34105$.

(b) **(i)** We consider two cases.

- **Case 1:** First, consider $n = k + 1$. There exists a partition of X such that $k - 1$ subsets each contain one object and the remaining subset, say S', contains two objects. S' can be partitioned into two subsets with one element each. So, X is the ancestor of $k+1$ partitions into $k+1$ non-empty subsets.

- **Case 2:** Next, consider $n > k+1$. There exists a partition of X such that $k-1$ subsets each contain one object and the remaining subset, say S'', contains $n-(k-1) = n-k+1$ objects. S'' can be split into two such that one subset contains one object and the other contains $n-k$ objects. There are $n-k+1$ ways to choose that one object. So, X is the ancestor of at least $n-k$ partitions into $k+1$ non-empty subsets.

The result follows.

(ii) We denote the partitions of the form X and the form Y to be

$$X_1, X_2, \ldots, X_{g(n,k)} \quad \text{and} \quad Y_1, Y_2, \ldots, Y_{g(n,k+1)} \quad \text{respectively.}$$

Also, let $d(X_i)$ and $a(Y_j)$ denote the following sets:

$$d(X_i) = \{\text{form } Y \text{ descendants that } X_i \text{ has}\} \quad \text{and} \quad a(Y_j) = \big\{\text{form } X \text{ ancestors that } Y_j \text{ has}\big\}$$

For any Y_j, its ancestors are the product of merging any 2 of its $k+1$ subsets. That is to say, for all j,

$$\big|a(Y_j)\big| = \binom{k+1}{2}.$$

We now prove the inequality.

$$\begin{aligned}
\text{RHS} &= \binom{k+1}{2} g(n,k+1) \\
&= \sum_{j=1}^{S(n,k+1)} \big|a(Y_j)\big| \\
&= \Big|\Big\{\text{number of tuples } (X_i, Y_j) \text{ where } X_i \text{ is an ancestor of } Y_j\Big\}\Big| \\
&= \sum_{i=1}^{S(n,k)} |d(X_i)| \\
&= \sum_{i=1}^{S(n,k)} (n-k) \\
&= (n-k)S(n,k) \\
&= \text{LHS}
\end{aligned}$$

so the inequality holds.

Next, we prove that equality holds if and only if $n = k+1$.
Suppose $n = k+1$. Then, for any X_i, there will only be 1 subset with 2 elements and the rest will all have 1 element. So, X_i only has 1 descendant, implying that $|d(X_i)| = n-(n-1) = 1$ for all $1 \le i \le g(n,k)$.

Remark for Question 1: For **(bii)**, suppose $n = k+1$, one can deduce that the inequality becomes an inequality very easily by using the same argument as given to deduce that

$S(k+1,k) = \binom{k+1}{2}$. However, if we are given the inequality and wish to prove that equality implies $n = k+1$, it is impossible to use the known recurrence relation for the Stirling numbers of the second kind.

Question 2

(a) Consider showing $x^2+y^2+z^2-3xyz=0$. Using the given substitutions, we have

$$\begin{aligned} x^2+y^2+z^2-3xyz &= a^2+(3ab-c)^2+b^2-3ab(3ab-c) \\ &= a^2+9a^2b^2-6abc+c^2+b^2-9a^2b^2+3abc \\ &= a^2+b^2+c^2-3abc \\ &= 0 \quad \text{since } (x,y,z)=(a,b,c) \text{ satisfies the equation} \end{aligned}$$

(b) Setting $a=1$, $b=1$ and $c=1$, we see that $3ab-c=2$, so $(x,y,z)=(1,2,1)$ is another solution.
Next, set $x=1$, $z=2$ and $y=3(1)(2)-1=5$, so $(x,y,z)=(1,5,2)$ is another solution.
Lastly, set $x=1$, $z=5$ and $y=3(1)(5)-2=13$, so $(x,y,z)=(1,13,5)$ is another solution.

(c) Let P_n be the proposition that

$$1+F_{2n+1}^2+F_{2n-1}^2=3F_{2n+1}F_{2n-1}$$

for all positive integers n.

When $n=1$, the LHS evaluates to $1+F_3^2+F_1^2=1+4+1=6$, whereas the RHS evaluates to $3F_3F_1=6$.
Hence, P_1 is true.
Suppose P_k is true for some positive integer k. That is, $1+F_{2k+1}^2+F_{2k-1}^2=3F_{2k+1}F_{2k-1}$.
To prove P_{k+1} is true, we need to show that $1+F_{2k+3}^2+F_{2k+1}^2=3F_{2k+3}F_{2k+1}$.
We apply **(a)** to the induction hypothesis to obtain

$$1+(3F_{2k+1}-F_{2k-1})^2+F_{2k+1}^2=3(3F_{2k+1}-F_{2k-1})F_{2k+1} \ (1).$$

Thus,

$$\begin{aligned} 1+F_{2k+3}^2+F_{2k+1}^2-3F_{2k+3}F_{2k+1} &= 1+(F_{2k+2}+F_{2k+1})^2+F_{2k+1}^2-3(F_{2k+2}+F_{2k+1})F_{2k+1} \\ &= 1+(2F_{2k+1}+F_{2k})^2+F_{2k+1}^2-3(2F_{2k+1}+F_{2k})F_{2k+1} \\ &= 1+(3F_{2k+1}+F_{2k-1})^2+F_{2k+1}^2-3(3F_{2k+1}+F_{2k-1})F_{2k+1} \\ &= 0 \text{ by } (1) \end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n by induction.

Remark for Question 2: The Diophantine equation in **(a)** is known as Markov's equation.

Question 3

(a) Write $t = an + p$, where $n \in \mathbb{Z}$ and $a > p \geq 0$. Then,

$$\begin{aligned}
\int_0^a \left\lfloor \frac{x+t}{a} \right\rfloor dx &= \int_0^a \left\lfloor \frac{x+p}{a} + n \right\rfloor dx \\
&= \int_0^a \left(n + \left\lfloor \frac{x+p}{a} \right\rfloor \right) dx \quad \text{since } n \in \mathbb{Z} \\
&= an + \int_0^{a-p} \left\lfloor \frac{x+p}{a} \right\rfloor dx + \int_{a-p}^a \left\lfloor \frac{x+p}{a} \right\rfloor dx \\
&= an + \int_0^{a-p} 0 \, dx + \int_{a-p}^a 1 \, dx \\
&= an + p = t
\end{aligned}$$

(b) **(i)** Motivated by **(a)**, consider the substitution $x = abn' + p'$, where $n' \in \mathbb{Z}$ and $ab > p \geq 0$.

The LHS becomes

$$\begin{aligned}
\left\lfloor \frac{\lfloor bn' + \frac{p}{a} \rfloor}{b} \right\rfloor &= \left\lfloor \frac{bn' + \lfloor \frac{p}{a} \rfloor}{b} \right\rfloor \quad \text{since } n' \in \mathbb{Z} \\
&= \left\lfloor n' + \frac{1}{b} \left\lfloor \frac{p}{a} \right\rfloor \right\rfloor \\
&= n' \quad \text{since } 0 \leq \frac{p'}{a} < b
\end{aligned}$$

We now justify that the RHS is also n'. The RHS can be written as

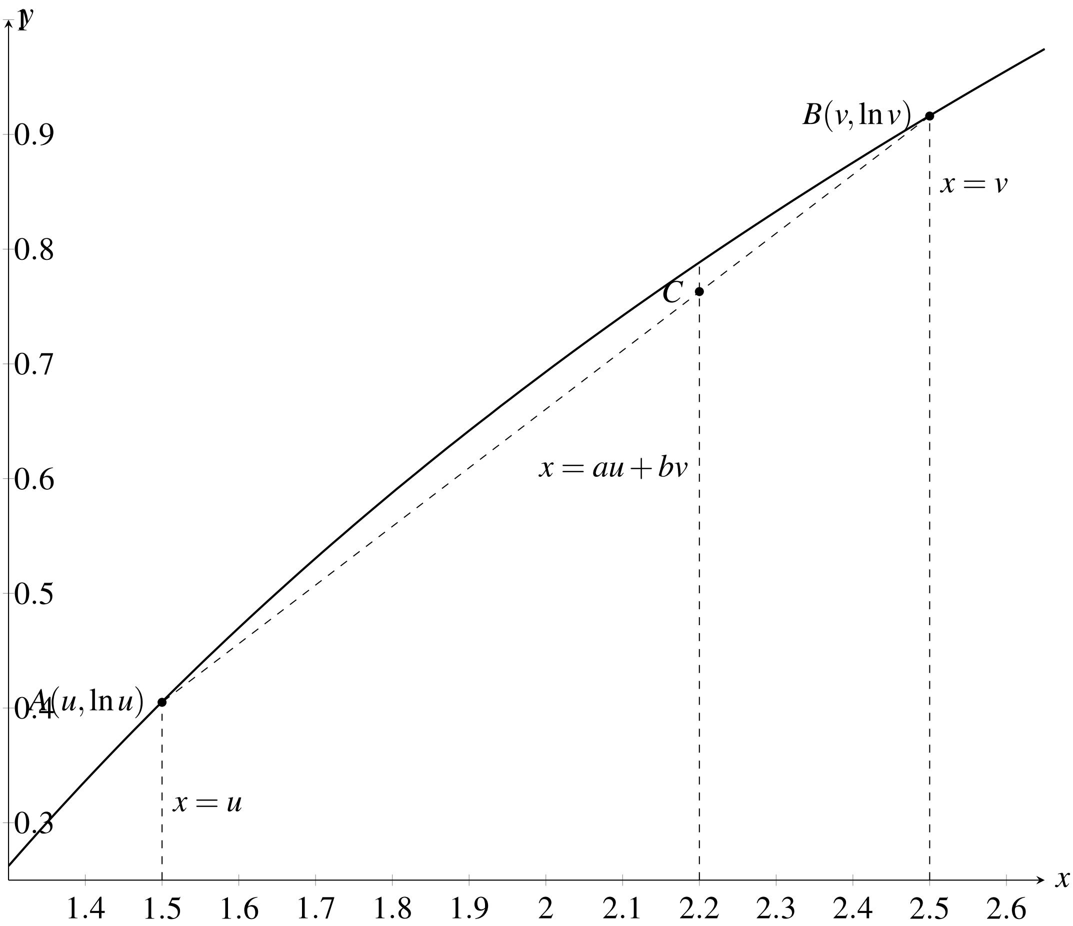
$$\begin{aligned}
\left\lfloor \frac{abn' + p'}{ab} \right\rfloor &= \left\lfloor n' + \frac{p'}{ab} \right\rfloor \quad \text{since } n \in \mathbb{Z} \\
&= n' + \left\lfloor \frac{p'}{ab} \right\rfloor \\
&= n'
\end{aligned}$$

(ii) Using the substitution for x in **(i)**, we have

$$\begin{aligned}
\int_0^{ab} (fg(x) - gf(x)) \, dx &= \int_0^{ab} \left\lfloor \frac{\lfloor \frac{x+b}{a} \rfloor + a}{b} \right\rfloor - \left\lfloor \frac{\lfloor \frac{x+a}{b} \rfloor + b}{a} \right\rfloor dx \\
&= \int_0^{ab} \left\lfloor \frac{\left\lfloor \frac{x+a^2+b}{a} \right\rfloor}{b} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{x+a+b^2}{b} \right\rfloor}{a} \right\rfloor dx \\
&= \int_0^{ab} \left\lfloor \frac{x+a^2+b}{ab} \right\rfloor - \left\lfloor \frac{x+a+b^2}{ab} \right\rfloor dx \quad \text{by } \textbf{(i)} \\
&= \left(a^2+b\right) - \left(a+b^2\right) \quad \text{by } \textbf{(a)} \\
&= a^2 - b^2 - a + b
\end{aligned}$$

Question 4

(a) We consider a sketch of the graph of $y = \ln x$. Without loss of generality, assume $u \le v$.



Since $a + b = 1$, where $a, b > 0$, $au + bv \in (u, v)$. This is because $y = \ln x$ is concave down for $x > 0$.

We first find the gradient of the line segment joining A and B. Consider the fact that $m_{AC} = m_{CB}$, so

$$\begin{aligned}
\frac{y - \ln u}{au + bv - u} &= \frac{y - \ln v}{au + bv - v} \\
auy + bvy - vy - au\ln u - bv\ln u + v\ln u &= auy + bvy - uy - au\ln v - bv\ln v + u\ln v \\
y\,(au + bv - v - au - bv + u) &= u\ln v - au\ln v - bv\ln v + au\ln u + bv\ln u - v\ln u \\
y &= \frac{(au + bv - v)\ln u + (u - au - bv)\ln v}{u - v} \\
&= a\ln u + b\ln v \quad \text{since } a + b = 1
\end{aligned}$$

So the y-coordinate of C is $a\ln u + b\ln v$, which is less than $\ln(au + bv)$.
As $a\ln u + b\ln v = \ln\left(u^a v^b\right)$ and $\ln x$ is an increasing function, the result follows.
Equality holds if and only if $u = v$.

(b) **(i)** Let $x_n = n\left(G_n - A_n\right)$. We shall prove that $x_{n+1} \leq x_n$. In other words, we can show that $x_{n+1} - x_n \leq 0$.
First, note that $-(n+1)A_{n+1} + nA_n = -a_{n+1}$.

$$\begin{aligned} x_{n+1} - x_n &= (n+1)\,G_{n+1} - (n+1)\,A_{n+1} - nG_n + nA_n \\ &= (n+1)\,(a_1a_2\ldots a_na_{n+1})^{\frac{1}{n+1}} - n(a_1a_2\ldots a_n)^{\frac{1}{n}} - a_{n+1} \end{aligned}$$

The second equality follows because $-(n+1)A_{n+1} + nA_n = -a_{n+1}$. So, the above expression simplifies as follows:

$$\begin{aligned} &(n+1)\left[(a_1a_2\ldots a_n)^{\frac{1}{n}}\right]^{\frac{n}{n+1}} a_{n+1}^{\frac{1}{n+1}} - n\,(a_1a_2\ldots a_n)^{\frac{1}{n}} - a_{n+1} \\ &\leq (n+1)\left[\frac{n}{n+1}\,(a_1a_2\ldots a_n)^{\frac{1}{n}} + \frac{1}{n+1}a_{n+1}\right] - n\,(a_1a_2\ldots a_n)^{\frac{1}{n}} - a_{n+1} \quad \text{by } \textbf{(a)} \\ &= 0 \end{aligned}$$

(ii) We can define $a_n = a_1a_2\ldots a_{n-1}$ for all $n \geq 4$. Since $a_{n-1} = a_1a_2\ldots a_{n-2}$, we have $a_n = a_{n-1}^2$.

Remark for Question 4: In **(bii)**, the sequence grows very rapidly. a_{13} has 925 digits, whereas a_{14} has 1850 digits.

Question 5

(a) **(i)** The number of ways to arrange the m married couples and s single people in a line is $(m+s)!$. We then multiply this by 2^m because within each of the m married couples, the husband and wife can swap positions.

(ii) For arrangements in a line, if the first and last persons form a married couple, then they must be seated together in the dining hall. However, this scenario is not accounted for when working with line arrangements.

(b) Define a k-vertex to be a vertex that is chosen to form our k-gon; a k^*-vertex is defined otherwise. This setup is now equivalent to distributing n vertices into k k-vertices and $n-k$ k^*-vertices, where the k-vertices are not adjacent.

We consider two cases.

- **Case 1:** Without loss of generality, suppose vertex 1 is a k-vertex, then the other two vertices are k^*-vertices.
Subsequently, insert the $n-k-2$ k^*-vertices. There are now $n-k-2+1 = n-k-1$ slots between the $n-k-2$ k^*-vertices. We can insert the $k-1$ k^*-vertices such that no two k^*-vertices are adjacent in $\binom{n-k-1}{k-1}$ ways.
- **Case 2:** Again without loss of generality, suppose vertex 1 is a k^*-vertex. Then, insert the $n-k-1$ k^*-vertices so that we have $n-k-1+1 = n-k$ slots within the k^*-vertices. We then insert the k k^*-vertices such that no two k^*-vertices are adjacent in $\binom{n-k}{k}$ ways.

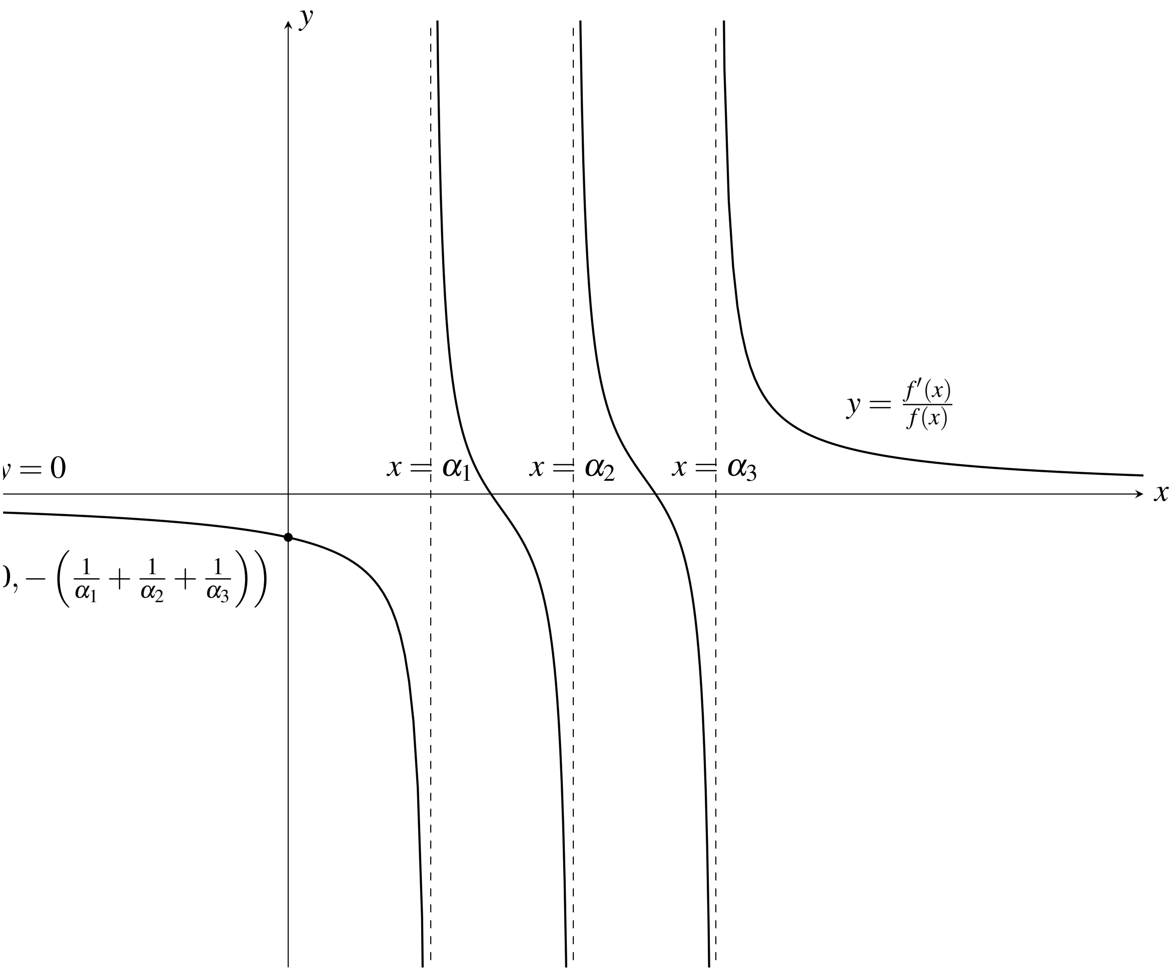
The total number of k-gons is $\binom{n-k-1}{k-1} + \binom{n-k}{k}$.

Question 6

(a) We have $f(x) = A(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$, where $A \neq 0$ is a constant. The sight of $f'(x)/f(x)$ prompts us to consider the derivative of $\ln(f(x))$.

$$\begin{aligned} \ln(f(x)) &= \ln A + \sum_{i=1}^{3} \ln(x-\alpha_i) \\ \frac{f'(x)}{f(x)} &= \sum_{i=1}^{3} \frac{1}{x-\alpha_i} \quad \text{by differentiating both sides with respect to } x \end{aligned}$$

(b) Consider the graph of $y = \frac{f'(x)}{f(x)}$. Without loss of generality, assume that $0 \leq \alpha_1 < \alpha_2 < \alpha_3$.



For the equation $f(x) - rf'(x) = 0$, we have to consider two cases.

- **Case 1:** Suppose $r = 0$, then $f(x) = 0$. Based on the preamble, $f(x) = 0$ has three distinct roots, $\alpha_1, \alpha_2, \alpha_3$, so the result follows.

- **Case 2:** Suppose $r \neq 0$. We can then rewrite the equation as $\frac{f'(x)}{f(x)} = \frac{1}{r}$. Any horizontal line $y = \frac{1}{r}$ intersects the graph at three distinct points, and the result follows.

(c) By **(b)**, $f(x) - \alpha_1 f'(x) = 0$ is a cubic equation with 3 distinct real roots.
Applying the result in **(b)** again, we have

$$\begin{aligned}[f(x) - \alpha_1 f'(x)] - \alpha_2[f'(x) - \alpha_1 f''(x)] &= 0 \\ f(x) - (\alpha_1 + \alpha_2)f'(x) + \alpha_1\alpha_2 f''(x) &= 0\end{aligned}$$

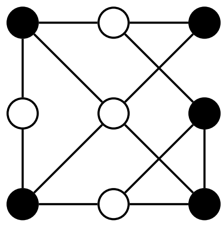
which is a cubic equation with 3 distinct real roots.
We apply **(b)** again to obtain

$$\begin{aligned}[f(x) - (\alpha_1 + \alpha_2)f'(x) + \alpha\alpha_2 f''(x)] - \alpha_3[f(x) - (\alpha_1 + \alpha_2)f'(x) + \alpha\alpha_2 f''(x)] &= 0 \\ f(x) - (\alpha + \alpha_2 + \alpha_3)f'(x) + (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)f''(x) - \alpha_1\alpha_2\alpha_3 f'''(x) &= 0 \\ f(x) + af'(x) + bf''(x) + cf'''(x) &= 0\end{aligned}$$

The last equality follows from the hint, which is essentially Vieta's formula. As such, we have obtained a cubic equation with 3 distinct real roots.

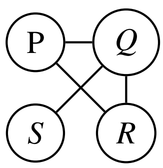
Question 7

(a) **(i)** Consider the following array which has 13 edges that link a shaded and an unshaded circle:



(ii) We first consider the case when $n = 4$ as shown in the second diagram. There are $4-1$ square blocks and $3^2 - 3$ arrowhead shapes. For some arbitrary n, there would be $n-1$ square blocks and $(n-1)^2 - (n-1) = n^2 - 3n + 2$ arrowhead shapes.

(iii) An arrowhead shape has 4 edges. Suppose on the contrary that 4 edges can link a shaded and an unshaded circle. Label the vertices as P, Q, R, S.



We consider two cases — when S is unshaded, and when S is shaded.

- **Case 1:** Suppose S is unshaded. Since S and Q share a common edge, then Q is shaded. So, P and R must be unshaded, which is a contradiction.

- **Case 2:** Suppose S is shaded. Similarly, Q is unshaded, implying that P and R are shaded, which is a contradiction as well.

As for a square block, at most 4 edges can link a shaded and an unshaded circle.

(b) **(i)** The 3×3 grid can be divided into three components which are the 4 corner squares, the 4 edge squares (but not including the corners), and the centre square. Denote the original sum by S and the final sum by S'. By symmetry, we only need to consider the cases when we shade either a corner square, an edge square or the centre square. We perform some casework.

- **Case 1:** Suppose we shade a corner square with a value of a. Then, S decreases by a, but the values of the centre square and two edge squares surrounding the corner square will increase by a total of a. So, $S' = S - a + a = S$.
- **Case 2:** Suppose we shade an edge square with a value of b, where b is the sum of the values of all the other squares. Then, S decreases by b. However, the total values in the two corner squares adjacent to it, as well as the centre squares, will increase by b, so $S' = S - b + b = S$.
- **Case 3:** Suppose we shade we shade the centre square with a value of c. Then, S decreases by c. However, S will increase by c concurrently too because each unshaded square other than the centre square will increment by some value and the total is c.

(ii) For an $n \times n$ grid,

- from **(aii)**, there are $n-1$ square blocks and from **(aiii)**, at most 4 edges in a square block can link a shaded and an unshaded circle;
- from **(aii)**, there are $n^2 - 3n + 2$ arrowhead shapes and from **(aiii)**, at most 3 edges in an arrowhead shape can link a shaded and an unshaded circle.

Thus, the maximum score is $4(n-1) + 3\left(n^2 - 3n + 2\right) = 3n^2 - 5n + 2$.

To achieve the maximum score, every square within a column must be consistently either shaded or unshaded, with adjacent columns alternating between these two states.

Question 8

(a) We see that

$$\begin{aligned}\left(r^2+s^2\right)\left(t^2+u^2\right)-(rt+su)^2 &= r^2t^2+s^2t^2+r^2u^2+s^2u^2-r^2t^2-2rstu-s^2u^2 \\ &= r^2u^2-2rstu+s^2t^2 \\ &= (ru-st)^2\end{aligned}$$

which is the square of $ru - st$.

(b) a^2 and b^2 must have opposite parities. That is to say, if a^2 is odd, then b^2 is even and vice versa.
Suppose a^2 is odd and b^2 is even. By contraposition, a is odd and b is even. So, there exists $\lambda, \mu \in \mathbb{Z}$ such that $a = 2\lambda + 1$ and $b = 2\mu$. To conclude,

$$\begin{aligned} n &= (2\lambda+1)^2 + (2\mu)^2 \quad \text{since } n = a^2+b^2 \\ &= 4\lambda^2 + 4\lambda + 4\mu^2 + 1 \end{aligned}$$

Choosing $k = \lambda^2 + \lambda + \mu$, the result follows.

(c) There exists $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ such that $m = \alpha^2 + \beta^2$ and $n = \gamma^2 + \delta^2$. Without loss of generality, suppose α and γ are odd, and β and δ are even. So,

$$\begin{aligned} 2mn &= 2\left(\alpha^2+\beta^2\right)\left(\gamma^2+\delta^2\right) \\ &= \left[(\alpha+\beta)^2 + (\alpha-\beta)^2\right]\left(\gamma^2+\delta^2\right) \\ &= \left[(\alpha+\beta)\delta + (\alpha-\beta)\gamma\right]^2 + \left[(\alpha+\beta)\gamma - (\alpha-\beta)\delta\right]^2 \quad \text{by } \textbf{(a)} \end{aligned}$$

which is the sum of two squares. We now show that

$$(\alpha+\beta)\delta + (\alpha-\beta)\gamma \quad \text{and} \quad (\alpha+\beta)\gamma - (\alpha-\beta)\delta$$

are odd. $\alpha+\beta$ and $\alpha-\beta$ are odd, so $(\alpha+\beta)\delta$ and $(\alpha-\beta)\delta$ are even, whereas $(\alpha-\beta)\gamma$ and $(\alpha+\beta)\gamma$ are odd. In each of the cases above, the sum of an odd integer and an even integer, so the resulting integer is even.

(d) Since the coefficients of $f(x)$ are real, by the conjugate root theorem, if $\lambda \in \mathbb{C}$ is a root of $f(x) = 0$, then λ^* is also a root of $f(x) = 0$, where λ^* is the complex conjugate of λ. We write

$$\begin{aligned} f(x) &= \text{product of all } (x-\lambda)(x-\lambda^*) \\ &= [\text{product of all } (x-\lambda)]\,[\text{product of all } (x-\lambda^*)] \\ &= [p(x)+iq(x)]\,[p(x)-iq(x)] \quad \text{where } p(x) \text{ and } q(x) \text{ are polynomials with real coefficients} \\ &= (p(x))^2 + (q(x))^2 \end{aligned}$$

so $f(x)$ is the sum of squares of two polynomials with real coefficients

Chapter 9

2023 Paper Solutions

Question 1

(a) Recall the Cauchy-Schwarz inequality, which states that for any real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, the inequality

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

holds. Set $x_i = a_i$ and $y_i = 1$ for all $1 \leq i \leq n$ so the inequality becomes

$$n\left(a_1^2 + a_2^2 + \ldots + a_n^2\right) \geq \left(a_1 + a_2 + \ldots + a_n\right)^2.$$

The result follows.

(b) It suffices to show that

$$\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x} \leq \sqrt{6}\sqrt{x+y+z}.$$

Using the Cauchy-Schwarz inequality mentioned in **(i)**, setting $x_1 = \sqrt{x+y}$, $x_2 = \sqrt{y+z}$, $x_3 = \sqrt{z+x}$ and $y_i = 1$ for all $1 \leq i \leq 3$, we have

$$\left(\sqrt{x+y} + \sqrt{y+z} + \sqrt{z+x}\right)^2 \leq 3\left(x+y+y+z+z+x\right).$$

As $x+y+y+z+z+x = 2\left(x+y+z\right)$, the result follows.

(c) Think of the equation as

$$\sqrt{\frac{x+3}{x+6}} + \sqrt{\frac{x+3}{x+6}} + \sqrt{\frac{6}{x+6}} = \sqrt{6}.$$

As such, consider $x+6 = x+y+z$, $x+y = x+3$, $x+z = x+3$ and $y+z = 6$, for which this implies $y = z = 3$. So, this deals with the equality case of **(ii)**, i.e. when

$$\sqrt{\frac{x+y}{x+y+z}} = \frac{\sqrt{6}}{3}.$$

So, $9\left(x+3\right) = 6\left(x+6\right)$, which implies $x = 3$.

Question 2

(a) Using $u = \frac{x}{y}$, we have

$$\frac{du}{dx} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}.$$

So,

$$\frac{du}{dx} = \frac{1}{y} - \frac{x}{y^2}\left(\frac{y}{x} - \frac{y^2}{x^3}\right) = \frac{1}{x^2}.$$

This implies that $u = -\frac{1}{x} + c$, where x is a constant. So, $\frac{x}{y} = -\frac{1}{x} + c$. It is easy to show that

$$y = \frac{x^2}{cx - 1}.$$

This is the equation of the curve C. Since C passes through (a,b), then $b = \frac{a^2}{ac-1}$. So, $c = \frac{a^2+b}{ab}$.

(b) Using long division, the equation of the curve C can be written as

$$y = \frac{x}{c} + \frac{1}{c^2} + \frac{1}{c^2\,(cx-1)}.$$

For C to have two asymptotes, we must have $c^2 \neq 0$, i.e. $a^2 \neq -b$. The vertical asymptote is $x = -\frac{1}{c} = -\frac{ab}{a^2+b}$ and the oblique asymptote is $y = \frac{x}{c} + \frac{1}{c^2} = \frac{abx}{a^2+b} + \left(\frac{ab}{a^2+b}\right)^2$.

Question 3

(a) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

(b) Starting with the RHS,

$$\sum_{j=0}^{n}\left(\binom{n}{j}\sum_{i=0}^{j}\binom{j}{i}\right) = \sum_{j=0}^{n}\binom{n}{j}2^j \quad \text{using } \textbf{(a)}$$

Note that the binomial expansion of $(2+x)^n$ is $\sum_{j=0}^{n}\binom{n}{j}2^j x^{n-j}$. Setting $x = 1$, the result follows.

(c) **(i)** If the divisor can be factorised into r primes (which are necessarily distinct) for $1 \le r \le k$, then the number of such divisors is $\binom{k}{r}$.

So, the total number of divisors is $\sum_{r=0}^{k}\binom{k}{r} = 2^k$.

(ii) $\mu(2) = -1; \mu(3) = -1; \mu(4) = 0; \mu(6) = 1; \mu(12) = 0$

(iii) We consider two cases.

- **Case 1:** If m is prime, then its only factors are 1 and m, so

$$\sum_{d|m}\mu(d) = \mu(1) + \mu(m) = 1 + (-1) = 0.$$

- **Case 2:** If m is composite, then $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, where $p_1, \dots, p_k$ are primes and $\alpha_1, \dots, \alpha_k \in \mathbb{Z}_{\geq 0}$. So,

$$\begin{aligned}\sum_{d|n} \mu(d) &= 1 + \sum_{r=1}^{k} \mu(p_r) + \sum_{i<j} \mu(p_i p_j) + \sum_{i<j<r} \mu(p_i p_j p_r) + \dots + \mu(p_1 p_2 \dots p_k) \\ &= 1 + k \cdot (-1) + \binom{k}{2} (-1)^2 + \dots + \binom{k}{k} (-1)^k \\ &= \sum_{r=0}^{k} \binom{k}{r} (-1)^r\end{aligned}$$

Consider $(1+x)^k = \sum_{r=0}^{k} \binom{k}{r} (-1)^r$. When we set $x = -1$, the RHS becomes the sum we wish to evaluate, while the LHS simplifies to zero.

Remark for Question 3: For **(c)**, μ is called the Möbius function.

Question 4

(a) $r_1 = 1$ as there is only one stone and that stone is coloured red; $r_2 = 0$ because if either stone is painted red, then the other cannot be painted red, otherwise, it will go against the condition that no two adjacent stones can be of the same colour.

$s_1 = 0$ as the stone cannot be painted red and not painted red concurrently; $s_2 = 3$ as there are three ways to paint the second stone, which are namely using white, yellow, or blue.

(b) Note that $r_3 = 3$ and $s_3 = 6$. So, $r_1 + s_1 = 1$, $r_2 + s_2 = 3$ and $r_3 + s_3 = 9$. So, we infer that $r_n + s_n = 3^{n-1}$.

r_{n+1} counts the number of ways to paint the stones such that the first stone is red and the $(n+1)^{\text{th}}$ stone is also red. As such, there are three choices to paint the n^{th} stone. So, the first n stones can be painted using s_n ways.

(c) Let P_n be the proposition that for all positive integers n,

$$r_n = \frac{3^{n-1} + 3(-1)^{n-1}}{4}$$

When $n = 1$, we have $r_1 = 1$ as obtained in **(a)**. The RHS also evaluates to 1, so P_1 is true.

Assume that P_k is true for some positive integer k. That is,

$$r_k = \frac{3^{k-1} + 3(-1)^{k-1}}{4}.$$

We wish to prove that P_{k+1} is true. That is,

$$r_{k+1} = \frac{3^k + 3(-1)^k}{4}.$$

From **(b)**, since $r_k + s_k = 3^{k-1}$ and $r_{k+1} = s_k$, then $r_k + r_{k+1} = 3^{k-1}$. As such,

$$\begin{aligned} r_{k+1} &= 3^{k-1} - r_k \\ &= 3^{k-1} - \frac{3^{k-1} + 3(-1)^{k-1}}{4} \quad \text{by induction hypothesis} \\ &= \frac{3\left(3^{k-1}\right) - 3(-1)^{k-1}}{4} = \frac{3^k + 3(-1)^k}{4} \end{aligned}$$

Since P_1 is true and P_k is true implies P_{k+1} is true, then P_n is true for all positive integers n.

(d) Suppose we colour the first stone red, then there are s_n ways to colour the remaining stones. By symmetry, the required answer is

$$\begin{aligned} 4s_n &= 4\left[3^{n-1} - \frac{3^{n-1} + 3(-1)^{n-1}}{4}\right] \\ &= 3^x - 3(-1)^{x-1} \end{aligned}$$

Remark for Question 4: For **(b)**, to justify that $r_n + s_n = 3^{n-1}$, note that $r_n + s_n$ counts the number of ways to place n stones on a line such that the first stone is red (and consequently, no restrictions on the last stone). Since there are $n-1$ positions to fill and there are 3 choices for each position, the result follows.

Question 5

(a) Possible remainders are 1 and 3.

(b) By Fermat's little theorem, as $z^{p-1} \equiv 1 \pmod{p}$, then $\left(z^2\right)^{(p-1)/2} \equiv 1 \pmod{p}$. So, $(-1)^{(p-1)/2} \equiv 1 \pmod{p}$. As such, $(p-1)/2$ must be even, so $p = 4k+1$, where $k \in \mathbb{Z}$. So, $p \equiv 1 \pmod{4}$. Equivalently, p is not congruent to 3 (mod 4).

(c) Possible remainders are 0, 1 and 4.

(d) **(i)** Suppose on the contrary that x is even. Then, there exists $m \in \mathbb{Z}$ such that $x = 2m$, so $y^2 = 8m^3 + 7$. A perfect square is 0 or 1 mod 4, so $8m^3 + 7 \equiv 3 \pmod{4}$, which is a contradiction.

(ii) We have $y^2 + 1 = x^3 + 8 = (x+2)(x^2 - 2x + 4) = (x+2)[(x-1)^2 + 3]$. It is clear that $(x-1)^2 + 3 \equiv 3 \pmod{4}$ and in fact, $(x-1)^2 + 3$ is of the form $4\alpha + 3$, where $\alpha \in \mathbb{Z}$. This is because x is odd implies $x-1$ is even, so we can write $x - 1 = 2\beta$, where $\beta \in \mathbb{Z}$. Hence, $(x-1)^2 + 3 = 4\beta^2 + 3$ (consequently, $\alpha = \beta^2$).

We claim that there exists a prime p such that $p \equiv 3 \pmod{4}$ such that p divides $y^2 + 1$. Note that $4\alpha + 3$ divides $y^2 + 1$ so there must exist some prime p of the form $4\gamma + 3$ that divides $4\alpha + 3$, where $\gamma \in \mathbb{Z}$. Suppose there does not exist such a prime. Then, the prime factors are of the form $4\gamma + 1$. Then, the product of the prime factors will be of the form 1 mod 4, which is not 3 mod 4. Thus, we reached a contradiction.

(iii) By **(ii)**, $y^2 \equiv -1 \pmod{p}$, where $p \equiv 3 \pmod{4}$. By **(b)**, p is not congruent to 3 mod 4, which is a contradiction.

Remark for Question 5: The equation $y^2 = x^3 + 7$ represents an elliptic curve, which has the general formula $y^2 = x^3 + ax + b$, where $4a^3 + 27b^2 \neq 0$. In particular, the equation in the question belongs to a class of elliptic curves known as Mordell curves, which has the general equation $y^2 = x^3 + n$, where n is a non-zero integer.

Elliptic curves play an important role in abstract algebra, particularly in tackling Fermat's last theorem.

Question 6

(a)

$$\begin{aligned}\int f(x)\ dx = \int e^{mx}\sin(mx)\ dx &= -\frac{1}{m}e^{mx}\cos(mx) + \int e^{mx}\cos(mx)\ dx \\ &= -\frac{1}{m}e^{mx}\cos(mx) + \frac{1}{m}e^{mx}\sin(mx) - \int e^{mx}\sin(mx)\ dx\end{aligned}$$

so it is clear that $\int f(x)\ dx = \dfrac{e^{mx}}{2m}[-\cos(mx) + \sin(mx)] + c$.

(b) Let $k = e^{-m\pi/2}$. Then,

$$\begin{aligned}\int f(x) f\left(x - \frac{\pi}{2}\right) dx &= k\int e^{2mx}\sin(mx)\sin\left(m\left(x - \frac{\pi}{2}\right)\right) dx \\ &= k\int e^{2mx}\sin(mx)\left[\sin(mx)\cos\left(\frac{m\pi}{2}\right) - \cos(mx)\sin\left(\frac{m\pi}{2}\right)\right] dx \\ &= \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}dx - \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}\cos(2mx)dx \\ &\quad - \frac{k}{2}\sin\left(\frac{m\pi}{2}\right)\int e^{2mx}\sin(2mx)dx \\ &= \frac{ke^{2mx}}{4m}\cos\left(\frac{m\pi}{2}\right) - \frac{k}{2}\cos\left(\frac{m\pi}{2}\right)\int e^{2mx}\cos(2mx)dx \\ &\quad - \frac{k}{2}\sin\left(\frac{m\pi}{2}\right)\int e^{2mx}\sin(2mx)dx.\end{aligned}$$

Note that

$$\begin{aligned}\int e^{mx}\cos(mx)\ dx &= e^{mx}\frac{1}{m}\sin(mx) - \int e^{mx}\sin(mx)\ dx \\ &= \frac{1}{m}e^{mx}\sin(mx) + \frac{1}{2m}e^{mx}\cos(mx) - \frac{1}{2m}e^{mx}\sin(mx) \text{ using } \mathbf{(a)} \\ &= \frac{e^{mx}}{2m}[\cos(mx) + \sin(mx)] + c\end{aligned}$$

so the original integral becomes

$$\begin{aligned}&\frac{ke^{2mx}}{4m}\cos\left(\frac{m\pi}{2}\right) - \frac{ke^{2mx}}{8m}\cos\left(\frac{m\pi}{2}\right)[\sin(2mx) + \cos(2mx)] - \frac{ke^{2mx}}{8m}\sin\left(\frac{m\pi}{2}\right)[\sin(2mx) - \cos(2m \\ =&\frac{ke^{2mx}}{8m}\left\{\cos\left(\frac{m\pi}{2}\right)[2 - \sin(2mx) - \cos(2mx)] - \sin\left(\frac{m\pi}{2}\right)[\sin(2mx) - \cos(2mx)]\right\}\end{aligned}$$

Recall the following as well:

$$\cos\left(\frac{m\pi}{2}\right) = \begin{cases} 0 & \text{if } m \text{ is odd;} \\ (-1)^{m/2} & \text{if } m \text{ is even} \end{cases} \text{ and } \sin\left(\frac{m\pi}{2}\right) = \begin{cases} (-1)^{(m-1)/2} & \text{if } m \text{ is odd;} \\ 0 & \text{if } m \text{ is even} \end{cases}$$

so if m is odd, then

$$\int f(x) f\left(x - \frac{\pi}{2}\right)\, dx = \frac{(-1)^{(m+1)/2} k e^{2mx}}{8m} \left[\sin(2mx) - \cos(2mx)\right] + c,$$

and if m is even, then

$$\int f(x) f\left(x - \frac{\pi}{2}\right)\, dx = \frac{(-1)^{m/2} k e^{2mx}}{8m} \left[2 - \sin(2mx) - \cos(2mx)\right] + c,$$

where c is a constant.

Question 7

(a) There are 4×3 squares of length 1 unit, 3×2 squares of length 2 units and 2×1 squares of 1 unit. The total number of squares is $12 + 6 + 2 = 20$.

(b) First, note that the largest square has length n units. There are $(m-1)(n-1)$ squares of length 1 unit, $(m-2)(n-2)$ squares of length 2 units and so on. So, there are $(m-k)(n-k)$ squares of length k units, where $1 \leq k \leq n$.
The total number of squares is

$$\begin{aligned} \sum_{k=1}^{n} (m-k)(n-k) &= \sum_{k=1}^{n} mn - (m+n) \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2 \\ &= mn^2 - \frac{n(n+1)(m+n)}{2} + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{1}{6} n \left[6m - 3(n+1)(n+m) + (n+1)(2n+1)\right] \\ &= \frac{1}{6} n \left(3mn + 1 - n^2 - 3m\right) \\ &= \frac{1}{6} n (n-1)(3m - n - 1) \end{aligned}$$

(c) Set

$$\frac{1}{6} n (n-1)(3m - n - 1) = 100.$$

Then, $n(n-1)(3m-n-1) = 600$.
We must have $n(n-1)$ to divide 600. The factors of 600 up to $\left\lfloor \sqrt{600} \right\rfloor$ are listed as follows:

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24$$

We shall test for $n = 2, 3, 4, 5, 6$.

- **Case 1:** If $n = 2$, then $m = 101$
- **Case 2:** If $n = 3$, then $m = 34\frac{2}{3} \notin \mathbb{N}$

- **Case 3:** If $n = 4$, then $m = 18\frac{1}{3} \notin \mathbb{N}$
- **Case 4:** If $n = 5$, then $m = 12$
- **Case 5:** If $n = 6$, then $m = 9$.

As such, the required pairs are $(m,n) = (101,2), (12,5)$ and $(9,6)$.

Question 8

(a) Note that the prime factorisation of 2400 is $2^5 \times 3 \times 5^2$.
Let $A, B, C \subseteq \{1, 2, \ldots, 2400\}$ be the sets of integers divisible by 2, 3, and 5 respectively. We wish to find $|A' \cap B' \cap C'|$, for which by de Morgan's law, is equal to $2400 - |A \cup B \cup C|$.
By the principle of inclusion and exclusion,

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \frac{2400}{2} + \frac{2400}{3} + \frac{2400}{5} - \frac{2400}{2 \times 3} - \frac{2400}{2 \times 5} - \frac{2400}{3 \times 5} + \frac{2400}{2 \times 3 \times 5} \\ &= 1760\end{aligned}$$

Hence, the required answer is $2400 - 1760 = 640$.

(b) **(i)** We have $N(a+b) = ab$. So, $(a-N)(b-N) = ab - N(a+b) + N^2 = N^2$.

(ii) Without loss of generality, assume that $a > b$. Then, we consider the following two cases:

- **Case 1:** Suppose $a - N = N$ and $b - N = N$. Then, $a = b = 2N$, which is a contradiction as $a > b$.
- **Case 2:** Suppose $a - N = N^2$ and $b - N = 1$. This is justified since $a = N^2 + N > N + 1 = b$.

So, there is only one way to express $\frac{1}{N}$ as a sum of two distinct unit fractions, which is $\frac{1}{N^2+N} + \frac{1}{N+1}$.

(i) Since $f(r) = 0$, then

$$r = \frac{B \pm \sqrt{B^2 - 4A}}{2}.$$

For r to be rational, we must have $B^2 - 4A$ to be a perfect square. Suppose there exists $k \in \mathbb{Z}_{\geq 0}$ such that $B^2 - 4A = k^2$. Hence, $(B+k)(B-k) = 4A$. So, $\frac{1}{2}(B+k)(B-k) = 2A$.
Observe that

$$\begin{aligned}r &= \frac{B \pm k}{\frac{1}{2}(B+k)(B-k)} \\ &= \frac{1}{\frac{B-k}{2}} \text{ or } \frac{1}{\frac{B+k}{2}}\end{aligned}$$

If B is even, then k is even; if B is odd, then k is odd.
So, the denominators of the unit fractions, $\frac{B-k}{2}$ and $\frac{B+k}{2}$, are positive integers.
The result follows.

(ii) By Vieta's formula, as the sum of roots is $\frac{7}{13}$, then $\frac{B}{A} = \frac{7}{13}$, so $7A = 13B$.
Since $\gcd(7,13) = 1$, there exists $k \in \mathbb{Z}$ such that $A = 13k$ and $B = 7k$.
We can rewrite the quadratic equation as $13kx^2 - 7kx + 1 = 0$, so

$$x = \frac{7k \pm \sqrt{49k^2 - 52k}}{26k}.$$

Since $49k^2 - 52k$ is a perfect square, there exists $m \in \mathbb{Z}$ such that $49k^2 - 52k = m^2$. By completing the square,

$$\left(k - \frac{26}{49}\right)^2 = \frac{49m^2 + 26^2}{49^2},$$

which means that $49m^2 + 26^2$ is also a perfect square. Then, there exists $\lambda \in \mathbb{Z}$ such that $49m^2 + 26^2 = \lambda^2$.
By the difference of squares formula, $(\lambda + 7m)(\lambda - 7m) = 26^2$.

- **Case 1:** Suppose $\lambda + 7m = 169$ and $\lambda - 7m = 4$. Then, $14m = 163$. One checks that $49m^2 + 26^2$ is not a perfect square.
- **Case 2:** Suppose $\lambda + 7m = 338$ and $\lambda - 7m = 2$. Then, $14m = 336$, so $49m^2 + 26^2 = 170^2$.

As such, $k = 4$ or $k = -\frac{144}{49}$.

For the sake of contradiction, suppose $k = -\frac{144}{49}$. Then, $49k^2 - 52k = 576 = 24^2$.
However,

$$\frac{7k - \sqrt{49k^2 - 52k}}{26k} = \frac{7k - 24}{26k} = \frac{7}{26} - \frac{12}{13k} < 0$$

which is a contradiction as $r_1, r_2 > 0$.

Thus, $k = 4$, so $49k^2 - 52k = 24^2$. This implies that

$$x = \frac{28 \pm 24}{104}.$$

Without loss of generality, set $r_1 = \frac{1}{2}$ and $r_2 = \frac{1}{26}$, so $A = 52$ and $B = 28$.

Chapter 10

2024 Paper Solutions

Question 1

(a) Since $(a-b)^2 \geq 0$, then $a^2 - 2ab + b^2 \geq 0$. Since $a, b > 0$, by cross multiplication, it suffices to prove that

$$(a+b)^2 \geq 4ab.$$

So, we have $a^2 + 2ab + b^2 \geq 4ab$, which simplifies to the desired inequality. Equality holds if and only if $a = b$.

(b) By the Cauchy-Schwarz inequality, we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \left(a\cdot\frac{1}{a}+b\cdot\frac{1}{b}+c\cdot\frac{1}{c}\right)^2 = 9.$$

Equality holds if and only if $a = b = c$.

(c) We first deduce the upper bound for

$$2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right).$$

By **(a)**, we have the following:

$$\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b} \text{ and } \frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c} \text{ and } \frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}.$$

Hence,

$$4\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \leq \frac{2}{a}+\frac{2}{b}+\frac{2}{c}.$$

Dividing both sides by 2 yields

$$2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

We then find the lower bound. From **(b)**, we replace a,b,c with $b+c, a+c, a+b$ respectively to obtain

$$2(a+b+c)\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right) \geq 9.$$

Dividing both sides by $a+b+c$ yields

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\right).$$

Question 2

(a) Let $P(n)$ denote the proposition that

$$\left(\sqrt{2}-1\right)^{2n-1} = A\sqrt{2}-B$$

for some positive integers A and B with $2A^2 - B^2 = 1$, where $n \in \mathbb{Z}^+$.

When $n = 1$, it is clear that we can choose $A = B = 1$, where A and B satisfy the equation $2A^2 - B^2 = 1$. So, the base case is true.

Assume that $P(k)$ holds for some $k \in \mathbb{Z}^+$. That is,

$$\left(\sqrt{2}-1\right)^{2k-1} = A\sqrt{2}-B \text{ where } 2A^2 - B^2 = 1.$$

We wish to prove that $P(k+1)$ is true, i.e.

$$\left(\sqrt{2}-1\right)^{2k+1} = A\sqrt{2}-B \text{ where } 2A^2 - B^2 = 1.$$

To see why this holds, we have

$$\begin{aligned}
\left(\sqrt{2}-1\right)^{2k+1} &= \left(\sqrt{2}-1\right)^{2k-1}\left(\sqrt{2}-1\right)^{2} \\
&= \left(A\sqrt{2}-B\right)\left(5-2\sqrt{2}\right) \\
&= (5A+2B)\sqrt{2}-(4A+5B)
\end{aligned}$$

So,

$$\begin{aligned}
2(5A+2B)^2-(4A+5B)^2 &= 2\left(25A^2+20AB+4B^2\right)-16A^2-40AB-25B^2 \\
&= 34A^2-17B^2 \\
&= 17\left(2A^2-B^2\right)
\end{aligned}$$

which evaluates to 0. Since $P(1)$ is true and $P(k)$ is true implies $P(k+1)$ is true, then by mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

(b) Multiplying both sides of the equation in **(a)** by $\sqrt{2}-1$ yields

$$\begin{aligned}\left(\sqrt{2}-1\right)^{2n} &= \left(A\sqrt{2}-B\right)\left(\sqrt{2}-1\right)\\ &= 2A+B-(A+B)\sqrt{2}\end{aligned}$$

Let $C=2A+B$ and $D=A+B$. So,

$$\begin{aligned}C^2-2D^2 &= (2A+B)^2-2(A+B)^2\\ &= 4A^2+4AB+B^2-2A^2-4AB-2B^2\\ &= 2A^2-B^2=1 \text{ by } \textbf{(a)}\end{aligned}$$

This implies $C^2-2D^2=1$.

(c) If n is odd, we replace n with $2n-1$ and use the result in **(a)**.
As such, $A\sqrt{2}=\sqrt{2A^2}$ and $B=\sqrt{B^2}$. Consequently, $N=2A^2$ and $N=B^2+1$.

If n is even, we replace n with $2n$ and use the result in **(b)**. As such, $N=C^2$ and $N=2D^2+1$.

In both cases, one checks that $2A^2-B^2=1$ and $C^2-2D^2=1$ are satisfied.

(d) Note that

$$\left(\sqrt{2}-1\right)^n\left(\sqrt{2}+1\right)^n=1.$$

This correlates with the equation $\left(\sqrt{N}-\sqrt{N-1}\right)\left(\sqrt{N}+\sqrt{N-1}\right)=1$, so by our construction of N in **(c)**, there also exists a positive integer N such that $\left(\sqrt{2}+1\right)^n=\sqrt{N}+\sqrt{N-1}$.

Question 3

(a) $f_n(0)=0$

(b) It would be easier if we write the terms of $f_n(x)$ explicitly.

$$\begin{aligned}f_n(x) &= \binom{n}{1}x-\binom{n}{2}\frac{x^2}{2}+\binom{n}{3}\frac{x^3}{3}+\ldots+(-1)^{n+1}\frac{x^n}{n}\\ f_n'(x) &= \binom{n}{1}-\binom{n}{2}x+\binom{n}{3}x^2+\ldots+(-1)^{n+1}x^{n-1}\\ xf_n'(x) &= \binom{n}{1}x-\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots+(-1)^{n+1}x^n\\ 1-xf_n'(x) &= 1-\binom{n}{1}x+\binom{n}{2}x^2-\binom{n}{3}x^3+\ldots+(-1)^n x^n \text{ since } (-1)^{n+2}=(-1)^n\\ &= \sum_{r=0}^{n}\binom{n}{r}(-x)^n\\ &= (1-x)^n\end{aligned}$$

The result follows.

(c) We have

$$\begin{aligned} f_n(x) - f_n(0) &= \int_0^x f_n'(t)\ dt \\ &= \int_0^x \frac{1-(1-t)^n}{t}\ dt \end{aligned}$$

Use the substitution $y = 1-t$ so $dy = -dt$. As such, the integral becomes

$$-\int_1^{1-x} \frac{1-y^n}{1-y}\ dy = \int_{1-x}^1 \frac{1-y^n}{1-y}\ dy.$$

(d) Putting $x = 1$ into **(c)** yields

$$\begin{aligned} f_n(1) - f_n(0) &= \int_0^1 \frac{1-y^n}{1-y}\ dy \\ f_n(1) &= \int_0^1 1 + y + y^2 + \ldots + y^{n-1}\ dy \\ &= \left[y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots + \frac{y^n}{n} \right]_0^1 \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \end{aligned}$$

(e) We have

$$\begin{aligned} f_{n+1}(1) - f_n(1) &= \sum_{r=1}^{n+1} (-1)^{r+1} \binom{n+1}{r} \frac{1}{r} - \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} \frac{1}{r} \\ &= (-1)^{n+2} \binom{n+1}{n+1} \frac{1}{n+1} + \sum_{r=1}^{n} \left[(-1)^{r+1} \binom{n+1}{r} - (-1)^{r+1} \binom{n}{r} \right] \frac{1}{r} \\ &= (-1)^n \frac{1}{n+1} + \sum_{r=1}^{n} (-1)^{r+1} \left[\binom{n+1}{r} - \binom{n}{r} \right] \frac{1}{r} \\ &= \frac{1}{n+1} + \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r-1} \frac{1}{r} \text{ since } n \text{ is even} \\ &= \frac{1}{n+1} + \sum_{r=0}^{n-1} (-1)^{r} \binom{n}{r} \frac{1}{r+1} \end{aligned}$$

By **(d)**, we know that

$$f_{n+1}(1) - f_n(1) = \frac{1}{n+1}$$

so

$$\sum_{r=0}^{n-1} (-1)^r \binom{n}{r} \frac{1}{r+1} = 0.$$

Multiplying each term of the sum by -1 yields the desired result, which is

$$\sum_{r=0}^{n-1} (-1)^{r+1} \binom{n}{r} \frac{1}{r+1} = 0.$$

(f) From our working in **(e)**, we have

$$\begin{aligned}
\frac{1}{n+1} &= (-1)^n \frac{1}{n+1} + \sum_{r=1}^{n} (-1)^{r+1} \left[\binom{n+1}{r} - \binom{n}{r} \right] \frac{1}{r} \\
&= -\frac{1}{n+1} + \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r-1} \frac{1}{r} \text{ since } n \text{ is odd} \\
\frac{2}{n+1} &= \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} \frac{1}{r+1}
\end{aligned}$$

$$\sum_{r=0}^{n-1} (-1)^{r+1} \binom{n}{r} \frac{1}{r+1} = -\frac{2}{n+1}$$

Question 4

(a) Using the substitution $y = \frac{x}{z}$, we have

$$\frac{dy}{dx} = \frac{z - x\dfrac{dz}{dx}}{z^2}.$$

Substituting this into the original differential equation yields

$$\begin{aligned}
\frac{1}{z} - \frac{x}{z^2}\frac{dz}{dx} &= \frac{1}{z} - \frac{2x^2}{z^2} e^{x^2} \\
\frac{x}{z^2}\frac{dz}{dx} &= \frac{2x^2}{z^2} e^{x^2} \\
\frac{dz}{dx} &= 2xe^{x^2} \\
z &= \int 2xe^{x^2}\, dx \\
&= e^{x^2} + c_1
\end{aligned}$$

So,

$$\frac{x}{y} = e^{x^2} + c_1.$$

When $x = 1$, we have $y = \frac{1}{c}$ so $c = e + c_1$, so $c_1 = c - e$. As such,

$$\begin{aligned}
\frac{x}{y} &= e^{x^2} + c - e \\
\frac{y}{x} &= \frac{1}{e^{x^2} + c - e} \\
y &= \frac{x}{e^{x^2} - e + c}
\end{aligned}$$

(b) **(i)** For the curve to have exactly two vertical asymptotes, the equation $e^{x^2} - e + c = 0$ must have exactly two real roots. So, $e^{x^2} = e - c$. Taking natural logarithm on both sides yields $x^2 = \ln(e-c)$. As such, the equations of the vertical asymptotes are $x = \pm\sqrt{\ln(e-c)}$. For the square root function to be defined, we must have $\ln(e-c) \geq 0$. Consequently, we must have $e - c \geq 1$, so $c \leq e - 1$.

(ii) We have

$$\frac{dy}{dx} = \frac{\left(e^{x^2} - e + c\right) - 2x^2 e^{x^2}}{\left(e^{x^2} - e + c\right)^2}.$$

Setting $\frac{dy}{dx} = 0$ yields

$$e^{x^2}\left(1 - 2x^2\right) - e + c = 0$$

Note that $e^{x^2} > 0$ for all values of x. So,

$$e^{x^2}\left(1 - 2x^2\right)\begin{cases} > 0 & \text{if } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}; \\ < 0 & \text{otherwise} \end{cases}$$

Also, note that the range of $e^{x^2}\left(1 - 2x^2\right)$ is $(-\infty, 1]$. As such, we must have $e^{x^2}\left(1 - 2x^2\right)$ to be sometimes positive. In order to achieve this, there must exist some x such that

$$e^{x^2}\left(1 - 2x^2\right) - e + c > 0 \text{ so } 1 - e + c > 0.$$

We conclude that $c > 1 - e$.

Question 5

(a) **(i)** Set $y = 1$ and the result follows.

(ii) Write

$$\frac{x^2 + 1}{x} = x + \frac{1}{x}$$

so setting $y = \frac{1}{x}$ yields the desired result.

(iii) Consider **(ii)**. By the AM-GM inequality, $x + \frac{1}{x} \geq 2$ for all $x > 0$. Also, for $x > 0$ the range of $x + \frac{1}{x}$ is $[2, \infty)$. As such for any $s \geq 2$, we have $f(s) = f(1)$. Next, for any $x > 0$, choose $n \in \mathbb{Z}$ such that $x + n \geq 2$. We can generalise the result in **(i)** to $f(x) = f(x + n)$. Since $x + n \geq 2$, then $f(x + n) = f(1)$. So, $f(x) = f(1)$ for all $x > 0$.

(b) Set $x = 0$ so $g\left(-y^2\right) = g\left(y^2\right)$. Since y^2 is an arbitrary variable, it follows that for any $u \in \mathbb{R}$, we have $g(-u) = g(u)$. As such, g is symmetrical about the y-axis, or rather, g is even.

From the original equation, set $y = -x$ so $g(0) = g\left(2x^2\right)$. Since this holds for any $x \in \mathbb{R}$, we can say that for any $t \in \mathbb{R}$, $g(t) = g(0)$. Since g is an even function, then $g(-t) = g(0)$. It follows that $g(x) = g(0)$ for all $x \in \mathbb{R}$, which is a constant function.

Question 6

(a) We wish to prove that for any $1 \leq i \leq n$, we have

$$\gcd(a_{n+1}, a_i) = 1.$$

Suppose this gcd is d. Then,

$$\begin{aligned} a_{n+1} &= (a+1)(2a+1) \\ &= 2a^2 + 3a + 1 \\ &= 2a_1^2 \ldots a_n^2 + 3a_1 \ldots a_n + 1 \end{aligned}$$

Since $d \mid a_{n+1}$, then $d \mid \left(2a_1^2 \ldots a_n^2 + 3a_1 \ldots a_n + 1\right)$. Also, given that $d \mid a_i$, then

$$d \mid 2a_1^2 \ldots a_n^2 \text{ and } d \mid 3a_1 \ldots a_n.$$

So,

$$d \mid \left(2a_1^2 \ldots a_n^2 + 3a_1 \ldots a_n\right).$$

As such, d divides the difference of $2a_1^2 \ldots a_n^2 + 3a_1 \ldots a_n + 1$ and $2a_1^2 \ldots a_n^2 + 3a_1 \ldots a_n$, so $d \mid 1$. Hence, $\gcd(a_{n+1}, a_i) = 1$, i.e. a_{n+1} is coprime to each of $a_1, \ldots, a_n$.

(b) We first list down the sequence of triangular numbers as follows:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, \ldots$$

So, $b_2 = 3$, $b_3 = 10$ and $b_4 = 91$.

Now, suppose on the contrary that the sequence $b_1, b_2, \ldots, b_n$ is finite. We wish to show that there exists a triangular number b_{n+1} such that b_{n+1} is coprime to each of $b_1, b_2, \ldots, b_n$. The idea is to use the result from **(a)** — trying to observe that $(a+1)(2a+1)$ is a triangular number too.

To see why, write

$$(a+1)(2a+1) = \frac{n(n+1)}{2}.$$

Then, $4a^2 + 6a + 2 = m^2 + m$ so it hints to write $m = 2a + k$, where $k \in \mathbb{Z}$ is to be determined. So, $4a^2 + 6a + 2 = 4a^2 + 4ak + k^2 + 2a + k$, so one checks that $k = 1$ works. As such, $m = 2a + 1$, i.e. we can define $b_{n+1} = (a+1)(2a+1)$.

By constructing such a b_{n+1}, we have shown that for any finite sequence $b_1, b_2, \ldots, b_n$ of triangular numbers, there exists another triangular number b_{n+1} that is coprime to all previous b_i. This contradicts the assumption that the sequence is finite.

Remark for Question 6: (b) of the question has some semblance to Problem 42 of W. Sierpiński's book on '250 Problems in Elementary Number Theory'. Moreover, b_n is known as the lexicographically earliest sequence of pairwise coprime triangular numbers.

Question 7

(a) Partition S into the following 25 sets:

$$\{\{n,n+5\}:n\in\{1,2,\ldots,5,11,12,\ldots,15,21,22,\ldots,25,31,32,\ldots,35,41,42,\ldots,45\}\}$$

There are 25 such subsets, whose union is S (by definition of a partition).

Next, let $D\subseteq S$ such that $|D|=27$. Since $27>25$, by the pigeonhole principle, at least one of these 25 subsets must contain at least two elements of D. If D contains two elements from the same subset, say $\{a,a+5\}$, then these two elements differ by exactly 5. Therefore, D must contain at least two numbers that differ by exactly 5.

(b) We partition S into 6 residue class modulo 6, so

$$S=\{1,7,\ldots,49\},\{2,8,\ldots,50\},\ldots,\{6,12,\ldots,48\}.$$

Each class has 9 elements. With 27 elements in D, by the pigeonhole principle, at least one residue class contains $\lceil\frac{27}{6}\rceil=5$ elements. Within that class, two elements must differ by 6 as they belong to an arithmetic progression with common difference 6.

However, D does not necessarily contain two numbers that differ by exactly 7. To see why, consider

$$D=\{1,\ldots,7,15,\ldots,21,23,\ldots,29,37,\ldots,42,48,49,50\}$$

which has $4\cdot 6+3=27$ elements but does not satisfy the mentioned property.

(c) Consider the congruence classes

$$\begin{aligned}\{4n+1:n\in\mathbb{Z}_{\geq 0}\}&=\{1,5,9,\ldots,45,49\}\\\{4n+2:n\in\mathbb{Z}_{\geq 0}\}&=\{2,6,10,\ldots,46,50\}\\\{4n+3:n\in\mathbb{Z}_{\geq 0}\}&=\{3,7,11,\ldots,47\}\end{aligned}$$

Let the above sets be S_1,S_2,S_3 respectively. Then, we wish to find $|S_1|+|S_2|+|S_3|$, which is $13+13+12=38$.

(d) We partition S into residue classes based on their remainder when divided by 10, where each class contains 5 numbers. For example, one residue class is $\{1,11,21,31,41\}$. Note that two numbers sum to a multiple of 10 if their residues sum to 0 modulo 10. The pairs are $(0,0),(1,9),(2,8),\ldots,(5,5)$.

However, we need to avoid sums which are forbidden. For example, given the pair $(1,9)$, we either choose all numbers in the residue class $\{1,11,21,31,41\}$ or $\{9,19,29,39,49\}$. The same applies to the pairs $(2,8),(3,7),(4,6)$. Selecting one residue from each pair yields $4 \times 5 = 20$ numbers.

Also, for the case where the sum of residues is congruent to 0 modulo 10, i.e. $r + r \equiv 0 \pmod{10}$, since $0 + 0 \equiv 0 \pmod{10}$, we select at most one number from its residue class. Similarly, since $5 + 5 \equiv 0 \pmod{10}$, we select at most one number from its residue class.

The total numbers selected is $20 + 2 = 22$, so the maximum possible size of such a subset of S is 22.

Question 8

(a) To go from $(0,0)$ to (n,n), we need to traverse n steps east and n steps north, where the order does not matter. This is equivalent to arranging $2n$ objects in a line with n type A objects and n type B objects, with objects of each type taken to be identical. So, the number of ways is

$$\frac{(2n)!}{(n!)^2} = \binom{2n}{n}.$$

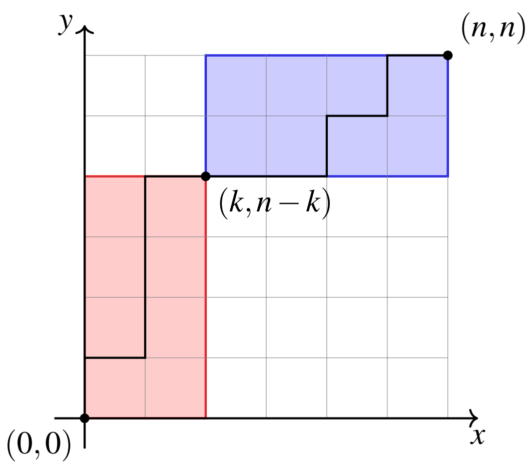
(b) We will only consider northeast paths. Similar to **(a)**, there are

$$\frac{(k+n-k)!}{k!\,(n-k)!} = \binom{n}{k}$$

ways to traverse from $(0,0)$ to $(k, n-k)$. Also, there are

$$\frac{(n-k+n-(n-k))!}{(n-k)!\,(n-(n-k))!} = \binom{n}{k}$$

ways to traverse from $(k, n-k)$ to (n,n)



Summing over all $0 \le k \le n$, there are

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{i=0}^{n} \binom{n}{i}^2$$

ways to traverse from $(0,0)$ to (n,n) that contain the point $(k,n-k)$. Since this sum also corresponds to the total number of northeast paths from $(0,0)$ to (n,n), the result follows.

(c) By **(b)**, the number of northeast paths from $(0,0)$ to (n,n) that contain (n,k) is

$$\sum_{k=1}^{n}\binom{n+k}{n}.$$

So, the number of northeast paths from $(0,0)$ to (n,n) that contain $(n,k-1)$ is

$$\sum_{k=1}^{n}\binom{n+k-1}{n}.$$

As such, the number of northeast paths from $(0,0)$ to (n,n) that contain (n,k) but do not contain $(n,k-1)$ is

$$\begin{aligned}\sum_{k=1}^{n}\left[\binom{n+k}{n}-\binom{n+k-1}{n}\right] &= \sum_{k=1}^{n}\binom{n+k-1}{n-1} \text{ by Pascal's identity}\\ &= \sum_{i=0}^{n-1}\binom{n+i}{n-1}\end{aligned}$$

On the other hand, applying the method of difference yields

$$\sum_{k=1}^{n}\left[\binom{n+k}{n}-\binom{n+k-1}{n}\right]=\binom{2n}{n}-1.$$

So,

$$\begin{aligned}\sum_{i=0}^{n-1}\binom{n+i}{n-1} &= \binom{2n}{n}-1\\ \binom{n}{n-1}+\binom{n+1}{n-1}+\ldots+\binom{2n-1}{n-1} &= \binom{2n}{n}-1\\ \binom{n-1}{n-1}+\binom{n}{n-1}+\binom{n+1}{n+1}n-1+\ldots+\binom{2n-2}{n-1} &= \binom{2n}{n}-\binom{2n-1}{n-1}\end{aligned}$$

To conclude,

$$\begin{aligned}\sum_{i=0}^{n-1}\binom{n-1+i}{n-1} &= \binom{2n}{n}-\binom{2n-1}{n-1}\\ &= \frac{(2n)!}{(n!)^2}-\frac{(2n-1)!}{n!\,(n-1)!}\\ &= \frac{(2n)!-n\,(2n-1)!}{(n!)^2}\\ &= \frac{(2n-1)!}{n!\,(n-1)!}\\ &= \binom{2n-1}{n}\end{aligned}$$

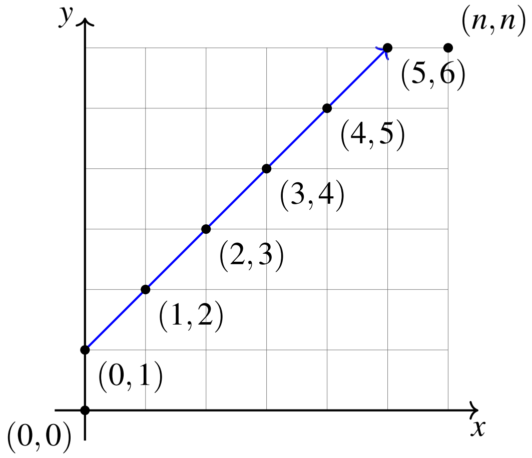
(d) **(i)** Define N and E to be one step north and one step east respectively. Identify the first step in the path that causes the path to cross the diagonal, i.e. the first step that reaches a point $(m, m+1)$. Replace this step N (the step that crosses the diagonal) with an E. After the replacement, continue the rest of the path unchanged.

This transformation converts a path from $(0,0)$ to (n,n) crossing the diagonal into a path from $(0,0)$ to $(n-1, n+1)$.

We then construct the reverse map. Start with a path from $(0,0)$ to $(n-1, n+1)$. Find the first occurrence of a step E such that the path immediately after this step is strictly above the diagonal, i.e. it goes from (m,m) to $(m+1, m+1)$). Replace this E with an N. After the replacement, continue the rest of the path unchanged.

This transformation reconstructs a path from $(0,0)$ to (n,n) crossing the diagonal.

This map is injective — each crossing point is uniquely identified in the forward map. Also, the reverse map precisely undoes the forward map by targeting the transformed step. Also the map is surjective — any path from $(0,0)$ to $(n-1, n+1)$ can be reached by starting with a diagonal-crossing path and applying the forward map. It follows that the map is bijective.



(ii) Let A denote the set of northeast paths from $(0,0)$ to (n,n) that cross the diagonal and B denote the set of northeast paths from $(0,0)$ to $(n-1, n+1)$. By **(i)**, the bijection principle implies $|A| = |B|$. We have

$$|B| = \frac{(n-1+n+1)!}{(n-1)!\,(n+1)!} = \frac{(2n)!}{(n-1)!\,(n+1)!} = \binom{2n}{n-1}.$$

So, the number of northeast paths from $(0,0)$ to (n,n) that do not cross the diagonal is

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{(n!)^2} - \frac{(2n)!}{(n-1)!\,(n+1)!} = \frac{1}{n+1}\binom{2n}{n}$$

Remark for Question 8: In **(dii)**, the sequence

$$\frac{1}{n+1}\binom{2n}{n}$$

that we obtained is known as the Catalan numbers. Such northeast paths are known as Dyck paths.

Chapter 11

2025 Paper Solutions

Question 1

(a) We see that the polynomial can be factorised as $3\left(x^6-1\right)-13x^2\left(x^2-1\right)$. We apply the difference of cubes formula to x^6-1, which yields

$$x^6-1=\left(x^2-1\right)\left(x^4+x^2+1\right).$$

Hence, we are finding the roots of the equation

$$3\left(x^2-1\right)\left(x^4+x^2+1\right)-13x^2\left(x^2-1\right)=0.$$

So,

$$\left(x^2-1\right)\left(3x^4+3x^2+3-13x^2\right)=0.$$

We see that $x=\pm 1$ are two roots. The two other roots can be obtained from solving $3x^4-10x^2+3=0$. Let $u=x^2$, so $3u^2-10u+3=0$. As such, either $u=\frac{1}{3}$ or $u=3$. Hence, $x=\pm\frac{1}{\sqrt{3}}$ or $x=\pm\sqrt{3}$, which yields six roots in total.

(b) We see that

$$\begin{aligned}3x^6-13x^4y^2+13x^2y^4-3y^6&=3\left(x^6-y^6\right)-13x^2y^2\left(x^2-y^2\right)\\&=3\left(x^2-y^2\right)\left(x^4+x^2y^2+y^4\right)-13x^2y^2\left(x^2-y^2\right)\end{aligned}$$

Hence,

$$3x^6-13x^4y^2+13x^2y^4-3y^6=\left(x^2-y^2\right)\left(3x^4-10x^2y^2+3y^4\right). \tag{11.1}$$

Setting $3x^6-13x^4y^2+13x^2y^4-3y^6=0$, we see that either $x^2-y^2=0$ or $3x^4-10x^2y^2+3y^4=0$. So, either $y=\pm x$ or $3x^2-10x^2y^2+3y^4=0$. In a similar fashion compared to **(a)**, we see that the equations of the lines are $y=\pm x$, $y=\pm\frac{1}{\sqrt{3}}x$, and $y=\pm\sqrt{3}x$.

(c) First, note that the region

$$\left\{(x,y)\in\mathbb{R}^2:|x|\leq\sqrt{3}\text{ and }|y|\leq 1\right\}$$

represents a rectangle in $\mathbb{R}^2$ bounded by the lines $x = \pm\sqrt{3}$ and $y = \pm 1$. For A, we need

$$3x^6 - 13x^4y^2 + 13x^2y^4 - 3y^6 \geq 0$$

which by (11.1) is equivalent to

$$\left(x^2 - y^2\right)\left(3x^4 - 10x^2y^2 + 3y^4\right) \geq 0.$$

So,

$$\left(x^2 - y^2\right)\left(3x^2 - y^2\right)\left(x^2 - 3y^2\right) \geq 0.$$

The critical points are $y = \pm x$, $y = \pm\frac{1}{\sqrt{3}}x$, and $y = \pm\sqrt{3}x$. By the test point method, one can deduce the solution to the inequality and sketch the shaded region A as shown in Figure 11.1. Note that the green lines have equations $y = \pm\sqrt{3}x$, the blue lines have equations $y = \pm x$, and the red lines have equations $y = \pm\frac{1}{\sqrt{3}}x$.

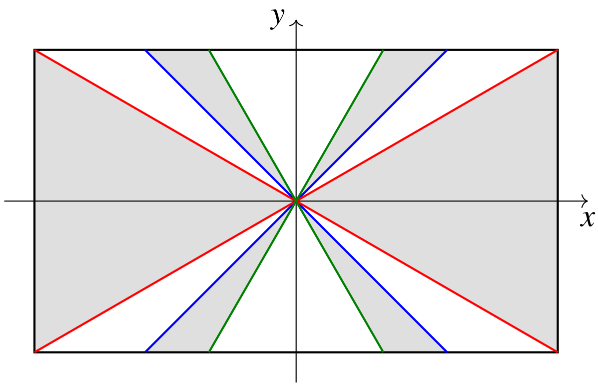


Figure 11.1: Region A

Figure 11.2 shows a sketch of region B, which can be regarded as the complement of region A in the rectangle bounded by the lines $x = \pm\sqrt{3}$ and $y = \pm 1$.

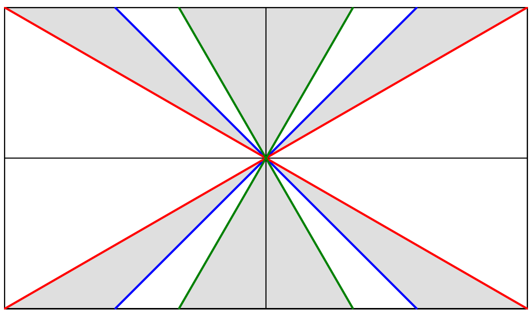


Figure 11.2: Region B

(d) The area of region A is

$$2 \cdot \frac{1}{2} \cdot 2 \cdot \sqrt{3} + 4 \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{\sqrt{3}}\right) \cdot 1 = 2\sqrt{3} + 2 - \frac{2}{\sqrt{3}}.$$

So, the area of region B is

$$2\sqrt{3}\cdot 2-\left(2\sqrt{3}+2-\frac{2}{\sqrt{3}}\right).$$

Hence, the ratio of the area of A to the area of B is 1.646 correct to 3 decimal places.

Question 2

(a) Let $y=ux$. Then,

$$\frac{dy}{dx}=u+x\frac{du}{dx}.$$

The differential equation becomes

$$x^2\left(u+x\frac{du}{dx}\right)=x^2+x\cdot ux+u^2x^2$$

so

$$x\frac{du}{dx}=1+u^2.$$

As such,

$$\int\frac{1}{1+u^2}\,du=\int\frac{1}{x}\,dx$$

which yields $\tan^{-1}u=\ln|x|+c$. When $x=1$, $y=0$ so $u=0$. This implies $c=0$. Hence, $\tan^{-1}u=\ln|x|$ so

$$\tan^{-1}\left(\frac{y}{x}\right)=\ln|x|.$$

To conclude, we have

$$y=x\tan(\ln|x|).$$

(b) We use the substitution $y=ux^2$.[1] So,

$$\frac{dy}{dx}=2ux+x^2\frac{du}{dx}.$$

The differential equation becomes

$$x^4\left(2ux+x^2\frac{du}{dx}\right)=x^4+2x^3\cdot ux^2+u^2x^4$$

so

$$x^2\frac{du}{dx}=1+u^2.$$

As such,

$$\int\frac{1}{1+u^2}\,du=\int\frac{1}{x^2}\,dx$$

which yields $\tan^{-1}u=-\frac{1}{x}+c$. When $x=1$, $y=0$ so $u=0$. This implies $c=0$. Hence, $\tan^{-1}u=-\frac{1}{x}$ so

$$\tan^{-1}\left(\frac{y}{x^2}\right)=-\frac{1}{x}.$$

To conclude, we have

$$y=x^2\tan\left(-\frac{1}{x}\right).$$

[1] The differential equation admits a similar structure to that in **(a)**. One may be tempted to use the substitution $y=ux$ but would realise that his/her attempt would be futile.

Question 3

(a) Using the AM-GM inequality for three variables,

$$\frac{x^2+\frac{y^2}{2}+\frac{y^2}{2}}{3} \geq \sqrt[3]{x^2\cdot\frac{y^2}{2}\cdot\frac{y^2}{2}} \quad \text{so} \quad \frac{x^2+y^2}{3} \geq \sqrt[3]{\frac{x^2y^4}{4}}.$$

Since $x^2+y^2=1$, then $\frac{4}{27} \geq x^2y^4$. Taking square roots on both sides yields the desired result. For equality to be achieved, we must have $x^2=\frac{y^2}{2}$ subjected to the constraint $x^2+y^2=1$. So, $3y^2=2$, which implies $y=\sqrt{\frac{2}{3}}$ since $y \geq 0$. Consequently, $x=\sqrt{\frac{1}{3}}$.

(b) With some trial and error, one should think of the AM-GM inequality applied to the following terms: two copies of x^2 and three copies of $\frac{y^2}{3}$. By the AM-GM inequality for five variables,

$$\frac{\frac{x^2}{2}+\frac{x^2}{2}+\frac{y^2}{3}+\frac{y^2}{3}+\frac{y^2}{3}}{5} \geq \sqrt[5]{\frac{x^2}{2}\cdot\frac{x^2}{2}\cdot\frac{y^2}{3}\cdot\frac{y^2}{3}\cdot\frac{y^2}{3}} \quad \text{so} \quad \frac{x^2+y^2}{5} \geq \sqrt[5]{\frac{x^4y^6}{108}}.$$

Since $x^2+y^2=1$, raising both sides to the fifth power yields

$$\left(\frac{1}{5}\right)^5 \geq \frac{x^4y^6}{108}.$$

Multiplying both sides by 108, we have

$$x^4y^6 \leq \frac{108}{5^5} \quad \text{so} \quad x^2y^3 \leq \frac{6\sqrt{3}}{25\sqrt{5}}.$$

For equality to be attained, we must have $\frac{x^2}{2}=\frac{y^2}{3}$ so $x^2=\frac{2}{3}y^2$. As such, $y^2=\frac{3}{5}$ which implies $y=\sqrt{\frac{3}{5}}$ and $x=\sqrt{\frac{2}{5}}$.

(c) With some careful analysis of **(a)** and **(b)**, in order to apply the AM-GM inequality, we need to *break* x^m into m copies of $\frac{x^2}{m}$ and y^n into n copies of $\frac{y^2}{n}$. By the AM-GM inequality for $m+n$ variables,

$$\frac{\frac{x^2}{m}+\ldots+\frac{x^2}{m}+\frac{y^2}{n}+\ldots+\frac{y^2}{n}}{m+n} \geq \sqrt[m+n]{\frac{x^2}{m}\cdot\ldots\cdot\frac{x^2}{m}\cdot\frac{y^2}{n}\cdot\ldots\cdot\frac{y^2}{n}}$$

so

$$\frac{x^2+y^2}{m+n} \geq \sqrt[m+n]{\frac{x^{2m}y^{2n}}{m^mn^n}}.$$

This implies

$$x^my^n \leq \sqrt{\frac{m^mn^n}{(m+n)^{m+n}}}$$

so we have obtained the maximum possible value of x^my^n. For equality to hold, we must have $\frac{x^2}{m}=\frac{y^2}{n}$ so $y^2=\frac{n}{m}x^2$. As such, $x^2\left(1+\frac{n}{m}\right)=1$ so $x=\sqrt{\frac{m}{m+n}}$ and $y=\sqrt{\frac{n}{m+n}}$.

Question 4

(a) **(i)** For the selected balls to be of the same colour, they must both be either white or black. Hence, the required probability is

$$\frac{w}{100} \cdot \frac{w-1}{99} + \frac{100-w}{100} \cdot \frac{99-w}{99} = \frac{1}{9900}\left[w\left(w-1\right)+\left(100-w\right)\left(99-w\right)\right].$$

Upon expansion, this yields

$$\frac{w^2-100w+4950}{4950}.$$

(ii) Let

$$f\left(w\right) = \frac{w^2-100w+4950}{4950}.$$

By completing the square on the numerator, we see that

$$f\left(w\right) = \frac{\left(w-50\right)^2+2450}{4950}.$$

The value of w that minimises the probability in **(i)** is 50. The minimum probability is $f\left(50\right) = \frac{49}{99}$.

(b) **(i)** We check the net change in the total number of balls for each possible pair. As such, we perform casework when selecting two balls at random.

- **Case 1:** If we have two white balls, we remove them and add 1 black ball so the total changes by $-2+1=-1$.
- **Case 2:** If we have either a (white, black) or (black, white), we remove 1 black ball and replace the white ball so the total changes by -1.
- **Case 3:** If we have two black balls, we remove 1 black ball and replace 1 black ball so the total changes by -1.

In all three cases, every step reduces the total number of balls by exactly 1. We started with 100 balls, so after 99 such steps there must be exactly 1 ball left.

(ii) We shall examine how the number of white balls changes over time. For a (white, white) pair, the number of white balls decreases by 2 so the parity is unchanged. For a (white, black) or (black, white) pair, the number of white balls is unchanged. Lastly, if we have a (black, black) pair, the number of white balls remains unchanged too. Thus, the parity of the number of white balls stays the same throughout the process. Initially, $W=45$ which is odd. At the end, only one ball remains so the last remaining ball is white with probability 1.

(c) **(i)** Suppose bag 1 has b blue balls and $n-b$ red balls. Then, bag 2 has $n-b$ blue balls and b red balls. The probability that the selected ball is blue is

$$\frac{1}{2} \cdot \frac{b}{n} + \frac{1}{2} \cdot \frac{n-b}{n} = \frac{b}{2n} + \frac{n-b}{2n} = \frac{1}{2}.$$

(ii) We must give a particular way to put the balls into the bags. One choice is as follows. Suppose bag 1 has 1 blue ball and 0 red balls. Then, bag 2 has $n-1$ blue balls and n red balls. So,

$$P(\text{blue} \mid \text{bag } 1) = 1 \quad \text{and} \quad P(\text{blue} \mid \text{bag } 2) = \frac{n-1}{2n-1}.$$

Hence (by what is known as the law of total probability),

$$P(\text{blue}) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{n-1}{2n-1} = \frac{1}{2}\left(1 + \frac{n-1}{2n-1}\right).$$

By using a graphing calculator or L'Hôpital's rule, one can see that $P(\text{blue})$ approaches $\frac{3}{4}$ as $n \to \infty$.

Question 5

(a) **(i)** Using integration by parts,

$$I_n = [(\ln t)^n \cdot t]_1^x - \int_1^x t \cdot n(\ln t)^{n-1} \cdot \frac{1}{t}\, dt = x(\ln x)^n - nI_{n-1}.$$

(ii) We have

$$I_n = xf_n(\ln x) + (-1)^{n+1} n! \quad \text{and} \quad I_{n-1} = xf_{n-1}(\ln x) + (-1)^n (n-1)!.$$

Substituting these into the recurrence relation in **(i)**, we have

$$xf_n(\ln x) + (-1)^{n+1} n! = x(\ln x)^n - n\left[xf_{n-1}(\ln x) + (-1)^n (n-1)!\right]$$

so

$$xf_n(\ln x) + (-1)^{n+1} n! = x(\ln x)^n - nxf_{n-1}(\ln x) + (-1)^{n+1} n!.$$

As such,

$$f_n(\ln x) + nf_{n-1}(\ln x) = (\ln x)^n.$$

Replacing $\ln x$ with x yields $f_n(x) + nf_{n-1}(x) = x^n$. So,

$$\begin{aligned} f_n(x) &= x^n - nf_{n-1}(x) \\ &= x^n - nx^{n-1} + n(n-1)f_{n-2}(x) \\ &= x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)f_{n-3}(x) \end{aligned}$$

One can continue this pattern and eventually deduce that

$$f_n(x) = \sum_{k=0}^{n} (-1)^{n-k} \cdot \frac{n!}{k!} x^k$$

which is a polynomial of degree n.

(iii) We see that

$$\begin{aligned} 1-nf_{n-1}(1) &= 1-n\sum_{k=0}^{n-1}(-1)^{n-1-k}\cdot\frac{(n-1)!}{k!} \\ &= 1-\sum_{k=0}^{n-1}(-1)^{n-1-k}\cdot\frac{n!}{k!} \\ &= \sum_{k=0}^{n}(-1)^{n-k}\cdot\frac{n!}{k!} \end{aligned}$$

which is equal to $f_n(1)$.

(b) **(i)** We have

$$J_n - J_{n+1} = \int_1^e \left[(\ln x)^n - (\ln x)^{n+1}\right]\,dx = \int_1^e (\ln x)^n(1-\ln x)\,dx.$$

If $n=0$, then

$$J_n - J_{n+1} = \int_1^e 1-\ln x\,dx.$$

Since $1-\ln x > 0$ on $(1,e)$, then the result is proven for $n=0$. If $n \geq 0$, then note that $(\ln x)^n > 0$ and $1-\ln x > 0$ so the integrand is the product of two positive functions, which is positive. The result follows.

(ii) Recognise that J_n is just I_n evaluated at $x=e$. Letting $x=e$ in **(ii)**, we have

$$J_n = I_n(e) = ef_n(1) + (-1)^{n+1}n!. \tag{11.2}$$

(iii) It suffices to prove that

$$\frac{n!}{f_n(1)} < e \quad \text{and} \quad e < -\frac{(n+1)!}{f_{n+1}(1)}.$$

respectively for all even $n \geq 2$. We refer to these as the *lower and upper bounds* respectively. We first prove the lower bound. From (11.2), we have

$$J_n = ef_n(1) + (-1)^{n+1}n!.$$

Since $n \geq 2$ is even, then

$$J_n = ef_n(1) - n!.$$

By definition of J_n, as $(\ln x)^n \geq 0$ on $[1,e]$, then $J_n > 0$. So, $ef_n(1) > n!$. We would need to justify that division by $f_n(1)$ does not flip the inequality sign. We will do so as we prove the upper bound concurrently.

To prove the upper bound, we have $J_{n+1} > 0$ so

$$ef_{n+1}(1) > -(n+1)!. \tag{11.3}$$

We now examine the sign of $f_{n+1}(1)$. From **(aiii)**, because $f_k(1) = 1 - kf_{k-1}(1)$, it follows by mathematical induction that for all $m \geq 1$, we have

$$f_{2m}(1) > 0 \quad \text{and} \quad f_{2m+1}(1) < 0.$$

Since $n \geq 2$ is even, then $f_{n+1}(1) < 0$. Dividing both sides of the inequality in (11.3) by $f_{n+1}(1)$ and flipping the sign, we obtain the upper bound. Consequently, we obtain the lower bound too.

Question 6

(a) By definition, $Z(121)$ denotes the set of Fibonacci numbers in the Zeckendorf representation of 121. So, $Z(121) = \{3, 8, 21, 89\}$.

(b) Let m be a positive integer. Let F_k denote the largest Fibonacci number that is $\leq m$. The idea is to replace m by $m - F_k$ and repeat while the remainder is > 0. We wish to prove that the process terminates and that the final sum is a sum of distinct non-consecutive Fibonacci numbers.

To prove that the process terminates, at each stage, we subtract a positive integer $F_k \geq 1$ from the current remainder, so the remainder is a strictly decreasing sequence of non-negative integers. Such a sequence cannot be infinite, otherwise it would contradict the well-ordering principle. Hence, the remainder eventually becomes 0 and the process stops.

Next, we prove that the Fibonacci numbers are distinct. Suppose at some step we choose F_k from a remainder r. Then, $F_k \leq r < F_{k+1}$ as F_{k+1} is the next Fibonacci number and is strictly larger than the remainder. The new remainder is

$$r' = r - F_k < F_{k+1} - F_k = F_{k-1}.$$

So, every subsequent remainder is $< F_{k-1}$ and hence $< F_k$. As such, we can never choose F_k again. So, all chosen Fibonacci numbers are distinct. Lastly, because $r < F_{k-1}$, it follows that no two chosen Fibonacci numbers are consecutive in the sequence.

(c) Let S denote a finite set of non-consecutive Fibonacci numbers with greatest element F_n. Let $\alpha = |S|$ denote the number of elements in S. We shall prove by induction on $\alpha \in \mathbb{N}$ that the sum of all the elements of S is $< F_{n+1}$.

For the base case $\alpha = 1$, we only have one Fibonacci number F_n, which is trivially $< F_{n+1}$. As such, the base case holds.

For the inductive step, suppose that the proposition holds for some $\alpha \geq 1$. That is, whenever S is a finite set of non-consecutive Fibonacci numbers with $|S| = \alpha$ and greatest element F_m, we have

$$\sum_{x \in S} x < F_{m+1}.$$

Let T be a finite set of non-consecutive Fibonacci numbers with $|T| = \alpha + 1$ and greatest element F_n. Since the elements of T are non-consecutive, $F_{n-1} \notin T$. Define $S = T \setminus \{F_n\}$ so that $|S| = \alpha$. Now, all the elements of S are among $F_1, \ldots, F_{n-2}$. Let the greatest element of S be F_k, where $k \leq n - 2$. By the inductive hypothesis applied to S,

$$\sum_{x \in S} x < F_{k+1} \leq F_{n-1}.$$

Therefore,
$$\sum_{x\in T} x = F_n + \sum_{x\in S} x < F_n + F_{n-1} = F_{n+1}.$$
Thus the result holds for sets with $\alpha+1$ elements. By strong induction, the sum of the elements of any finite set of non-consecutive Fibonacci numbers with greatest element F_n is strictly less than F_{n+1}.

(d) Suppose on the contrary that a positive integer m has more than one Zeckendorf representation. Then, we can write
$$m = \sum_{x\in S} x = \sum_{y\in T} y.$$
Here, S and T are sets of non-consecutive Fibonacci numbers and $S \neq T$. Let the greatest Fibonacci number in S be F_p and in T be F_q. Without loss of generality, assume that $p > q$, so $F_p \in S$ and every element of T is at most $F_q \leq F_{p-1}$. Since the greatest element of S is F_p, by **(c)**,
$$m = \sum_{x\in S} x < F_{p+1}.$$
In a similar fashion, because the greatest element of T is at most F_{p-1}, then
$$m = \sum_{y\in T} y < F_p.$$
Since $F_p \in S$, then $m \geq F_p$ which shows that $F_p \leq m < F_p$. This is a contradiction, and we conclude that every positive integer has at most one Zeckendorf representation.

(e) From **(b)**, we showed that every positive integer has a Zeckendorf representation (existence) and from **(d)**, we showed that every positive integer has at most one Zeckendorf representation (uniqueness). The result follows.

(f) Suppose $Z(m)$ has k elements. The idea is to take the k smallest non-consecutive Fibonacci numbers $F_1, F_3, F_5, \ldots, F_{2k-1}$ so the minimal sum for a representation with k terms is
$$T_k = F_1 + F_3 + F_5 + \ldots + F_{2k-1}.$$
One checks that $T_1 = 1$, $T_2 = 4$, $T_3 = 12$, $T_4 = 33$, $T_5 = 88$, and $T_6 = 232$. So, the maximum possible size of $Z(m)$ for $m < 144$ is 5. As such, we must find all $m < 144$ whose Zeckendorf representation has 5 Fibonacci numbers.

Clearly, one possible representation is
$$88 = 55 + 21 + 8 + 3 + 1.$$
Since 89 is the smallest Fibonacci number that is > 88, to form a Zeckendorf representation that has 5 Fibonacci numbers given that one of them is 89, we focus on $F_1, \ldots, F_8$ excluding $F_9 = 55$ since F_9 and $F_{10} = 89$ are consecutive. Hence, we

must pick 4 non-consecutive indices from $\{1,\ldots,8\}$.

Eventually, one can enumerate all possible representations, which are

$$\begin{aligned}
88 &= 55+21+8+3+1 \\
122 &= 89+21+8+3+1 \\
135 &= 89+34+8+3+1 \\
140 &= 89+34+13+3+1 \\
142 &= 89+34+13+5+1 \\
143 &= 89+34+13+5+2
\end{aligned}$$

To conclude, the values of m are $88, 122, 135, 140, 142, 143$.

Chapter 12

2017 Specimen Paper Solutions

Question 1

(a) Using the Cauchy-Schwarz inequality, we have

$$\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right]\left[\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+\left(\frac{x}{y}\right)^2\right] \geq \left[\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)+\left(\frac{y}{z}\right)\left(\frac{z}{x}\right)+\left(\frac{z}{x}\right)\left(\frac{x}{y}\right)\right]^2$$

so

$$\left[\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2\right]^2 \geq \left(\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)^2$$

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2 \geq \frac{x}{z}+\frac{y}{x}+\frac{z}{y}$$

so we have proven the upper bound for $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$.

Next, using the AM-GM inequality, we have

$$\frac{x}{z}+\frac{y}{x}+\frac{z}{y} \geq 3\sqrt[3]{\left(\frac{x}{z}\right)\left(\frac{y}{x}\right)\left(\frac{z}{y}\right)} = 3$$

so we have proven the lower bound for $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$.

(b) **(i)** By definition of the scalar product, for any two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we have $\mathbf{ab} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. Since $|\cos\theta| \leq 1$, then $\mathbf{ab} \leq |\mathbf{a}||\mathbf{b}|$, which implies that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \leq \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right|\left|\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right|$$

and the result follows. For equality to hold, we must have $a_i = kb_i$ for all $1 \leq i \leq 3$ and some $k \in \mathbb{R}\backslash\{0\}$.

(ii) Using the Cauchy-Schwarz inequality,

$$\left[\left(\frac{x}{\sqrt{y+z}}\right)^2+\left(\frac{y}{\sqrt{z+x}}\right)^2+\left(\frac{z}{\sqrt{x+y}}\right)^2\right]\left[\left(\sqrt{y+z}\right)^2+\left(\sqrt{z+x}\right)^2+\left(\sqrt{x+y}\right)^2\right]\geq(x+y+z)^2$$

So,

$$\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)(y+z+z+x+x+y)\geq(x+y+z)^2$$
$$2\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\geq x+y+z$$

and equality holds if and only if $x=y=z$.

Question 2

(i) We use the substitution $u=x^2$, which is motivated by the presence of the square root in the denominator and the fact that 4 and 9 are square numbers. Hence, the integral becomes

$$\int_2^3 \frac{x^2}{x-1}\,dx=\int_2^3 x+1+\frac{1}{x-1}\,dx=\ln 2+\frac{7}{2}.$$

(ii) Using the substitution $y=xu$, we have

$$\frac{dy}{dx}=x\frac{du}{dx}+u$$
$$\frac{1}{x}\frac{dy}{dx}=\frac{du}{dx}+\frac{u}{x}$$

so the differential equation becomes $\frac{du}{dx}=f(u)$.

(iii) Dividing both sides by x, we have

$$\frac{1}{x}\frac{dy}{dx}=\sqrt{\frac{x}{y}}-\frac{x}{y}+\frac{y}{x^2}$$

so

$$f\left(\frac{y}{x}\right)=\sqrt{\frac{x}{y}}-\frac{x}{y}.$$

As such,

$$f(u)=\frac{1}{\sqrt{u}}-\frac{1}{u}=\frac{\sqrt{u}-1}{u}.$$

The differential equation becomes $\frac{du}{dx}=\frac{\sqrt{u}-1}{u}$. From **(i)**, we have

$$\frac{2u\sqrt{u}+3u+6\sqrt{u}+6\ln|\sqrt{u}-1|}{3}=x+c, \tag{12.1}$$

where c is a constant. As the solution curve passes through $\left(\frac{1}{3},\frac{4}{3}\right)$, then $u=4$. Substituting these into (12.1) yields $c=13$. Hence,

$$2\frac{y}{x}\sqrt{\frac{y}{x}}+3\left(\frac{y}{x}\right)+6\sqrt{\frac{y}{x}}+6\ln\left|\sqrt{\frac{y}{x}}-1\right|=3x+39.$$

When $y=9x$, we have $60+6\ln 2=3x$, so $x=20+2\ln 2$, which is the required x-coordinate.

Question 3

(i) **(a)** Let $S = \{a, 2a, \dots (p-1)a\}$. For all $1 \leq i \leq p-1$, none of the $ia \in S$ is divisible by p because a is not divisible by p. Suppose $ai \equiv aj \pmod{p}$. Then, there exists $\lambda \in \mathbb{Z}$ such that $ai = \lambda p + aj$, so $a(i-j) = \lambda p$. However, p does not divide a so p must divide $i - j$. That is, $i \equiv j \pmod{p}$. As $1 \leq i, j \leq p-1$, then $i = j$ so all the elements in S are distinct. In $\bmod p$, the elements in S are a permutation of T, where $T = \{1, 2, \dots, p-1\}$.

(b) In $\bmod p$, the product of the elements in S is congruent to the product of the elements in T. That is,

$$a \cdot 2a \cdot 3a \cdot (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdot (p-1) \pmod{p}.$$

So, $a^{p-1} \equiv 1 \pmod{p}$.

(ii) By the binomial theorem,

$$\begin{aligned}(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\ &= x^5 + y^5 + 5k\end{aligned}$$

where $k \in \mathbb{Z}$. So, $x^5 + y^5 = (x+y)^5 - 5k$. As $x^5 + y^5 \equiv 0 \pmod{5}$, then $(x+y)^5 \equiv 0 \pmod{5}$.

We use the method of contraposition to prove that $x + y \equiv 0 \pmod{5}$. That is to say, given that $x + y$ is not a multiple of 5, we wish to prove that $(x+y)^5$ is also not a multiple of 5. We write $x + y = 5k + r$, where $1 \leq r \leq 4$. So,

$$(x+y)^5 = (5k+r)^5 = 3125k^5 + 3125k^4r + 1250k^3r^2 + 250k^2r^3 + 25kr^4 + r^5$$

so $(x+y)^5 \equiv r^5 \pmod{5}$ but because $1 \leq r \leq 4$, then r^5 is not a multiple of 5 and so we have proven that $x + y \equiv 0 \pmod{5}$. As such, there exists $\alpha \in \mathbb{Z}$ such that $x + y = 5\alpha$. Then, $x = 5\alpha - y$. Hence,

$$x^5 + y^5 = (5\alpha - y)^5 + y^5 = 3125\alpha^5 - 3125\alpha^4y + 1250\alpha^3y^2 - 250\alpha^2y^3 + 25\alpha y^4.$$

It follows that $x^5 + y^5$ is divisible by 25.

Remark for Question 3: This deals with a well-known result in number theory called Fermat's little theorem. It states that if $\gcd(a, p) = 1$ (i.e. a is not divisible by p), then $a^{p-1} \equiv 1 \pmod{p}$. An alternative representation says that for any integer a, $a^p \equiv a \pmod{p}$. Our method of proving Fermat's little theorem was using modulo inverse.

Question 4

(i) 5^n

(ii) **(a)** $B_1 = 5$ and $B_2 = 24$; B_2 can be calculated easily by considering the complement of the event 'never chooses Scrambled eggs on consecutive days' so $B_2 = 5^2 - 1$.

(b) We consider two cases.

- **Case 1 (Scrambled eggs on the 1st day):** On the 2nd day, she has 4 choices remaining. There would be no restrictions on what she has on the remaining $n-2$ days. This contributes to $4B_{n-2}$.
- **Case 2 (no Scrambled eggs on the 1st day):** On the 1st day, she has 4 choices. Thereafter, she has no restrictions on what she has on the remaining days. This contributes to $4B_{n-1}$.

Since the 2 cases are mutually exclusive, the result follows.

(c) Let P_k be the proposition that $B_{3k+1} \equiv 0 \pmod 5$ for all $k \in \mathbb{Z}_{\geq 0}$.
When $k = 0$, we have $B_1 = 5$, which is divisible by 5. So, P_0 is true.
Assume P_r is true for some $r \in \mathbb{Z}_{\geq 0}$. Then, $B_{3r+1} \equiv 0 \pmod 5$. We are required to show $B_{3r+4} \equiv 0 \pmod 5$. Using the relation in **(iib)**, as $B_k = 4B_{k-1} + 4B_{k-2}$, then

$$\begin{aligned} B_{3r+4} &= 4B_{3r+3} + 4B_{3r+2} \\ &= 4\left(4B_{3r+2} + 4B_{3r+1}\right) + 4B_{3r+2} \\ &= 20B_{3r+2} + 16B_{3r+1} \\ &\equiv 16B_{3r+1} \pmod 5 \\ &\equiv 0 \pmod 5 \quad \text{by induction hypothesis} \end{aligned}$$

Since P_0 is true and P_r is true implies P_{r+1} is true, by mathematical induction, P_k is true for all $k \in \mathbb{Z}_{\geq 0}$.

Question 5

(i) **(a)** Consider the graph of $y = x^p$ for $p < 0$ and $x > 0$ as shown in Figure 12.1 (we set $i = 2$ here but actually, i is arbitrary):

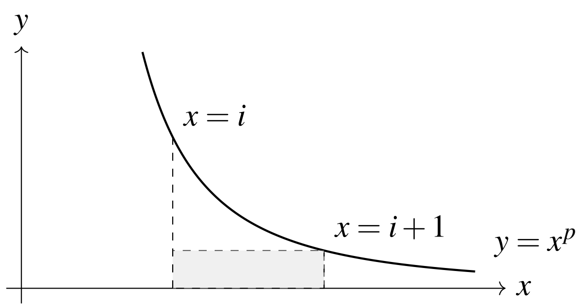


Figure 12.1:

$\displaystyle\int_i^{i+1} x^p \, dx$ denotes the area bounded by the curve, the x-axis and the ordinates $x = i$ and $x = i+1$. We construct the rectangle as shown in Figure 12.1 which has a base of 1 unit and a height of $(i+1)^p$. Its area is $(i+1)^p$ units^2, which is less than the given integral.

(b) It suffices to prove that

$$\int_i^{i+1} x^p\,dx < \frac{i^p+(i+1)^p}{2}.$$

Naturally, we would think of the right side of the inequality as the area of another figure other than a rectangle. Consider Figure 12.2:

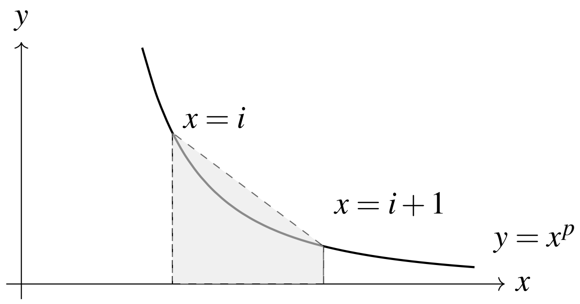


Figure 12.2:

We construct a trapezium bounded by the x-axis and the ordinates $x=i$ and $x=i+1$. Its area is $\frac{i^p+(i+1)^p}{2}$. The integral is less than the area of the trapezium and the result follows.

(ii) Using **(ia)**, we see that

$$(i+1)^p < \int_i^{i+1} x^p\,dx.$$

As such,

$$2^p+3^p+\ldots+n^p < \int_1^2 x^p\,dx + \int_2^3 x^p\,dx + \ldots + \int_{n-1}^n x^p\,dx.$$

So,

$$\sum_{k=1}^n k^p < 1 + \int_1^n x^p\,dx.$$

The required sum is $\sum_{k=1}^\infty k^p$ so as $n\to\infty$, we have

$$\sum_{k=1}^\infty k^p < 1 + \left[\frac{x^{p+1}}{p+1}\right]_0^\infty = \lim_{n\to\infty}\left(1+\frac{n^{p+1}-1}{p+1}\right) = 1-\frac{1}{p+1} = \frac{p}{1+p}.$$

(iii) In **(ii)**, we used **(ia)** to show that

$$2^p+3^p+\ldots+n^p < \int_1^n x^p\,dx. \tag{12.2}$$

Considering the integral on the right side of (12.2), we have

$$\int_1^n x^p\,dx = \frac{n^{p+1}-1}{p+1}.$$

Adding $1^p = 1$ to both sides, we establish an upper bound for $1^p+2^p+3^p+\ldots+n^p$.

Using **(ib)**, we have

$$\begin{aligned}
\int_i^{i+1} x^p\,dx &< \frac{i^p+(i+1)^p}{2} \\
\int_1^2 x^p\,dx + \int_2^3 x^p\,dx + \ldots + \int_{n-1}^n x^p\,dx &< \frac{1^p+2^p}{2} + \frac{2^p+3^p}{2} + \ldots + \frac{(n-1)^p+n^p}{2} \\
\int_1^n x^p\,dx &< \frac{1^p}{2} + \frac{n^p}{2} + \sum_{k=2}^{n-1} k^p \\
\frac{n^{p+1}-1}{p+1} &< \frac{1^p}{2} + \frac{n^p}{2} + \sum_{k=2}^{n-1} k^p \\
\frac{1^p+n^p}{2} + \frac{n^{p+1}-1}{p+1} &< \sum_{k=1}^{n} k^p
\end{aligned}$$

so we have established a lower bound for $1^p+2^p+3^p+\ldots+n^p$.

Therefore,

$$\begin{aligned}
\frac{1+n^p}{2n^{p+1}} + \frac{n^{p+1}-1}{n^{p+1}(p+1)} < \frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}} &< \frac{1}{n^{p+1}} + \frac{n^{p+1}-1}{n^{p+1}(p+1)} \\
\frac{1}{2n^{p+1}} + \frac{1}{n} + \frac{1}{p+1} - \frac{1}{n^{p+1}(p+1)} < \frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}} &< \frac{1}{n^{p+1}} + \frac{1}{p+1} - \frac{1}{n^{p+1}(p+1)}
\end{aligned}$$

As $p > -1$, then $p+1 > 0$. As $n \to \infty$ on both sides, by the squeeze theorem, the upper and lower bounds will tend to $\frac{1}{p+1}$. Therefore,

$$\lim_{n\to\infty}\left(\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}\right) = \frac{1}{p+1}.$$

For Question 5, there is a formula for the sum $1^p+2^p+\ldots+n^p$ which is known as Faulhaber's formula. It states that

$$\sum_{k=1}^n k^p = \frac{1}{p+1}\sum_{r=0}^{p}\binom{p+1}{r}B_r n^{p-r+1},$$

where B_r denotes the sequence of Bernoulli numbers.

Question 6

(i) Since f is continuous on $[0,0.4]$, differentiable on $(0,0.4)$, $f(0) = 1 > 0$ and $f(0.4) = -0.136 < 0$, by the intermediate value theorem, there exists a root in $(0,0.4)$.

Next, since f is continuous on $[0.4,2]$, differentiable on $(0.4,2)$, $f(0.4) = -0.136 < 0$ and $f(2) = 3 > 0$, by the intermediate value theorem, there exists a root in $(0.4,2)$.

Lastly, since f is continuous on $[-2,0]$, differentiable on $(-2,0)$, $f(-2) = -1 < 0$

and $f(0) = 1 > 0$, by the intermediate value theorem, then there exists a root in $(-2, 0)$.

The above shows that f has at least three distinct real roots. To show that there are only three distinct real roots, consider $f'(x) = 3\left(x^2 - 1\right)$ so f is strictly increasing for $x > 1$ and strictly decreasing for $x < -1$.

(ii) Note that

$$f(g(x)) = f\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^3 - 3\left(\frac{1}{1-x}\right) + 1 = -\frac{1 - 3x + x^3}{(1-x)^3}$$

so $g(\alpha), g(\beta)$ and $g(\gamma)$ are the roots of f. From **(i)**, $\alpha \in (-2, 0)$, $\beta \in (0, 0.4)$ and $\gamma \in (0.4, 2)$. We have $g(\gamma) < 0$, which implies that $g(\gamma) = \alpha$. Suppose on the contrary that $g(\beta) = \beta$. Then,

$$\frac{1}{1-\beta} = \beta.$$

That is, $\beta^2 - \beta + 1 = 0$. However, the roots of this equation are not real, which is a contradiction. As such, $g(\beta) = \gamma$, leaving us with $g(\alpha) = \beta$.

(iii) Write $h(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Then for $x = \alpha, \beta, \gamma$, we have

$$\begin{aligned}
ax^2 + bx + c &= \frac{1}{1-x} \\
ax^2(1-x) + bx(1-x) + c(1-x) - 1 &= 0 \\
ax^2 - ax^3 + bx - bx^2 + c - cx - 1 &= 0
\end{aligned}$$

As such,

$$-ax^3 + (a-b)x^2 + (b-c)x + c - 1 = 0. \tag{12.3}$$

Comparing (12.3) with $f(x)$, $a = -1$, $b = -1$ and $c = 2$. So, $h(x) = -x^2 - x + 2$.

For Question 6**(i)**, to put it more rigorously, the justification of the existence of a root is due to the intermediate value theorem. It states that given a continuous function f on an interval $[a, b]$ such that $f(a)$ and $f(b)$ have different polarities (i.e. either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$), then there exists some $c \in (a, b)$ such that $f(c) = 0$.

Question 7

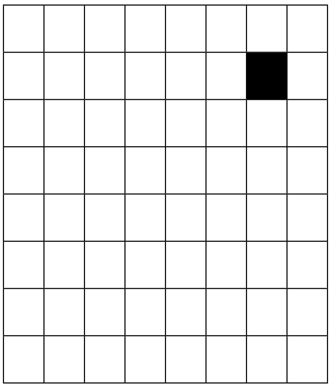
(i) Consider a square board with 4^n unit squares. Without a loss of generality, suppose a unit square in the 1st quadrant is covered. Then, consider the 4 unit squares at the centre. Cover all the squares except that in the 1st quadrant. As the board can be rotated in any direction, regardless of which unit square is originally covered, the result follows.

(ii) First, note that the area of the board is 4^n units^2, then the length must be 2^n units. Let P_n be the proposition that on a $2^n \times 2^n$ square board, if one unit square is initially covered, then the remaining unit squares can be covered by triominoes, and the total number of triominoes required is $\frac{1}{3}(4^n - 1)$ for all $n \in \mathbb{N}$.

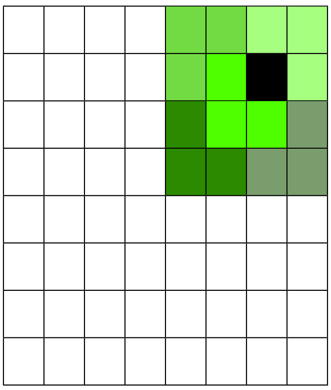
When $n = 1$, we have a 2×2 square board. If one unit square is initially covered, then we have 3 unit squares remaining. They are arranged in an L-shape. The remaining unit squares can be covered by triominoes. The total number of triominoes is 1. Hence, P_1 is true.

Assume that P_k is true for some $k \in \mathbb{N}$. That is, on a $2^k \times 2^k$ square board, if one unit square is initially covered, the remaining unit squares can be covered by $\frac{1}{3}\left(4^k - 1\right)$ triominoes. We wish to prove that P_{k+1} is true. That is, on a $2^{k+1} \times 2^{k+1}$ square board, if one unit square is initially covered, the remaining unit squares can be covered by $\frac{1}{3}\left(4^{k+1} - 1\right)$ triominoes.

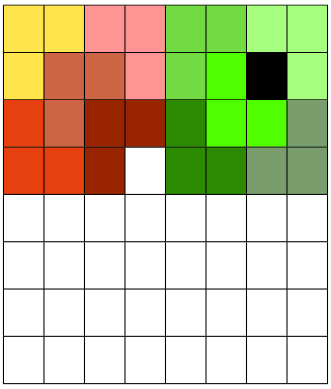
Consider a $2^{k+1} \times 2^{k+1}$ square board. Divide it into four $2^k \times 2^k$ square boards. Without a loss of generality, suppose a unit square in the 1st quadrant is initially covered as shown.



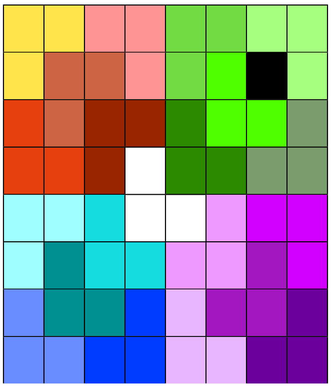
By the induction hypothesis, that $2^k \times 2^k$ board can be covered by $\frac{1}{3}\left(4^k - 1\right)$ triominoes.



For the $2^k \times 2^k$ board in the 2nd quadrant, we can cover it with $\frac{1}{3}\left(4^k - 1\right)$ triominoes such that a unit square remains in the bottom-right corner.



Repeat this process for the $2^k \times 2^k$ boards in the 3rd and 4th quadrants and do not occupy the top-right and top-left corners respectively. This can be covered with $2 \times \frac{1}{3}\left(4^k - 1\right)$ triominoes.



Finally, the 2×2 square board in the centre can be covered by one triomino.

We see that the total number of triominoes required is $\frac{4}{3}\left(4^k - 1\right) + 1 = \frac{1}{3}\left(4^{k+1} - 1\right)$. Since P_1 is true and P_k is true implies P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{N}$.

Question 8

(a) Let S_1 and S_2 be the following sets:

$$S_1 = \{x \in \mathbb{Z} : a \leq x \leq b\} \quad \text{and} \quad S_2 = \{x \in \mathbb{Z} : c \leq x \leq d\}$$

One can sketch a number line and come up with two cases.

- **Case 1:** Suppose $c \leq a \leq b \leq d$. Then, $S_1 \subseteq S_2$. Since $|S_1| \leq |S_2|$, then the number of integers x is $b - a + 1$.
- **Case 2:** Suppose $a \leq c \leq d \leq b$. Then, $S_2 \subseteq S_1$. Since $|S_2| \leq |S_1|$, the number of integers x is $d - c + 1$.

The result follows.

(b) Consider $x + y = n$ and $0 \leq y \leq b$. Then, $0 \leq n - x \leq b$, so $n - b \leq x \leq n$. Thus, the equation $x + y = n$ is restricted to the conditions $0 \leq x \leq a$ and $n - b \leq x \leq n$.

Since $a+b \geq n$, then $a \geq n-b$. Consider $x+y=n$ and $0 \leq x \leq a$. Then, $0 \leq n-y \leq a$, so $n-a \leq y \leq n$. Thus, the equation $x+y=n$ is restricted to the conditions $0 \leq y \leq b$ and $n-a \leq y \leq n$. The result follows.

(c) Let A, B and C denote the following sets:

$$A=\{x \in \mathbb{Z}_{\geq 0} : x > a\} \quad B=\{y \in \mathbb{Z}_{\geq 0} : y > a\} \quad \text{and} \quad C=\{z \in \mathbb{Z}_{\geq 0} : z > a\}$$

So,

$$\begin{aligned}\left|A' \cap B' \cap C'\right| &= |\xi| - |A \cup B \cup C| \quad \text{by de Morgan's law} \\ &= |\xi| - 3|A| + 3|A \cap B| - |A \cap B \cap C|\end{aligned}$$

where we used the principle of inclusion and exclusion in the second line. Note that $|A \cap B \cap B| = 0$. If the cardinality was positive, it would imply that $x+y+z > 3a$ but this contradicts the fact that $x+y+z \leq 3a$.

Hence,

$$|A| = \binom{n-a+1}{2}, \ |A \cap B| = \binom{n-2a}{2} \text{ and } |A \cap B \cap C| = 0.$$

Therefore,

$$\left|A \cap B' \cap C'\right| = \binom{n+2}{2} - 3\binom{n-a+1}{2} + 3\binom{n-2a}{2}.$$

Remark for Question 8: Here is an interactive solution to **(ii)**.

Chapter 13

2025 Specimen Paper Solutions

Question 1

(a) Since $y = x$, then $\frac{dy}{dx} = 1$, so the LHS of the differential equation becomes $x^2 + x^2 - x^2 - x^2$, which is zero.

(b) Letting $u = \frac{y}{x}$, we have

$$\frac{du}{dx} = \frac{1}{x^2}\left(x\frac{dy}{dx} - y\right).$$

The differential equation becomes

$$\frac{du}{dx} = \frac{xF(u) - y}{x^2} = \frac{F(u) - u}{x}.$$

The result follows by multiplying x on both sides of the equation.

(c) We have

$$\frac{dy}{dx} = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1.$$

Making reference to **(b)**, we see that $F\left(\frac{y}{x}\right) = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$. So, $F(u) = u^2 + u - 1$. The differential equation becomes

$$x\frac{du}{dx} = u^2 - 1.$$

So,

$$\int \frac{1}{u^2 - 1}\,du = \int \frac{1}{x}\,dx,$$

which implies that

$$\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| = \ln|x| + c,$$

where x is an arbitrary constant. When $x = 1$ and $y = 2$, we have $u = 2$, so substituting $(x, u) = (1, 2)$ into the equation of the above equation yields $c = -\frac{1}{2}\ln 3$.

Therefore,

$$
\begin{aligned}
\frac{1}{2}\ln\left|\frac{y-x}{y+x}\right| &= \ln|x| - \frac{1}{2}\ln 3 \\
\ln\left|\frac{3(y-x)}{y+x}\right| &= 2\ln|x| \\
\frac{3(y-x)}{y+x} &= x^2 \\
y &= \frac{x^3+3x}{3-x^2}
\end{aligned}
$$

which is the required equation of the curve.

Question 2

(a) We use integration by parts. So,

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{3}} \tan^n\theta \, d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \tan^2\theta \, d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \sec^2\theta \, d\theta - \int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \, d\theta \quad \text{since } \tan^2\theta = \sec^2\theta - 1 \\
&= \left[\tan^{n-1}\theta\right]_0^{\frac{\pi}{3}} - (n-2)\int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \sec^2\theta \, d\theta - I_{n-2} \\
&= 3^{\frac{n-1}{2}} - (n-2)\int_0^{\frac{\pi}{3}} \tan^n\theta \, d\theta - (n-2)\int_0^{\frac{\pi}{3}} \tan^{n-2}\theta \, d\theta - I_{n-2} \\
&= 3^{\frac{n-1}{2}} - (n-2)I_n - (n-2)I_{n-2} - I_{n-2} \\
(n-1)I_n &= 3^{\frac{n-1}{2}} - (n-1)I_{n-2}
\end{aligned}
$$

Dividing by $n-1$ yields the result.

(b) It is clear that $I_0 = \frac{\pi}{3}$.
Also,

$$
\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{3}} \tan\theta \, d\theta \\
&= \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta} \, d\theta \\
&= [\ln|\cos\theta|]_0^{\frac{\pi}{3}} \\
&= \ln 2
\end{aligned}
$$

So,

$$\begin{aligned}I_5 &= \frac{3^2}{4} - I_3 \\ &= \frac{3^2}{4} - \frac{3}{2} + I_1 \\ &= \frac{3^2}{4} - \frac{3}{2} + \ln 2 \\ &= \frac{3}{4} + \ln 2\end{aligned}$$

and

$$\begin{aligned}I_6 &= \frac{9\sqrt{3}}{5} - I_4 \\ &= \frac{9\sqrt{3}}{5} - \sqrt{3} + I_2 \\ &= \frac{9\sqrt{3}}{5} - \sqrt{3} + \sqrt{3} - I_0 \\ &= \frac{9\sqrt{3}}{5} - \frac{\pi}{3}\end{aligned}$$

Question 3

(a) **(i)** By the AM-GM inequality, we have

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2\right] \geq \sqrt{\left(\frac{x}{y}\right)^2\left(\frac{y}{z}\right)^2} = \frac{x}{z}.$$

(ii) Using the Cauchy-Schwarz inequality,

$$\begin{aligned}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2\right]\left[\left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + \left(\frac{x}{y}\right)^2\right] &\geq \left[\left(\frac{x}{y}\right)\left(\frac{y}{z}\right) + \left(\frac{y}{z}\right)\left(\frac{z}{x}\right) + \left(\frac{z}{x}\right)\left(\frac{x}{y}\right)\right]^2 \\ \left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2\right]^2 &\geq \left(\frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right)^2 \\ \left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 &\geq \frac{x}{z} + \frac{y}{x} + \frac{z}{y}\end{aligned}$$

so we have proven the upper bound for $\frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

Next, using the AM-GM inequality, we have

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} \geq 3\sqrt[3]{\left(\frac{x}{z}\right)\left(\frac{y}{x}\right)\left(\frac{z}{y}\right)} = 3$$

so we have proven the lower bound for $\frac{x}{z} + \frac{y}{x} + \frac{z}{y}$.

(b) **(i)** By definition of the scalar product, for any two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

we have $\mathbf{ab} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors. Since $|\cos\theta| \leq 1$, then $\mathbf{ab} \leq |\mathbf{a}||\mathbf{b}|$, which implies that

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \leq \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| \left|\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right|$$

and the result follows. For equality to hold, we must have $a_i = kb_i$ for all $1 \leq i \leq 3$ and some $k \in \mathbb{R}\setminus\{0\}$.

(ii) Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left[\left(\frac{x}{\sqrt{y+z}}\right)^2 + \left(\frac{y}{\sqrt{z+x}}\right)^2 + \left(\frac{z}{\sqrt{x+y}}\right)^2\right] \left[\left(\sqrt{y+z}\right)^2 + \left(\sqrt{z+x}\right)^2 + \left(\sqrt{x+y}\right)^2\right] &\geq (x+y+z \\ \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right)(y+z+z+x+x+y) &\geq (x+y+z \\ 2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) &\geq x+y+z \end{aligned}$$

and equality holds if and only if $x = y = z$.

Remark for Question 3: Other than **(ai)**, the other three parts are the same as Question 1 of the 2017 specimen paper.

Question 4

(i) Since f is continuous on $[0, 0.4]$, $f(0) = 1 > 0$ and $f(0.4) = -0.136 < 0$, then there exists a root in $(0, 0.4)$.
Next, since f is continuous on $[0.4, 2]$, $f(0.4) = -0.136 < 0$ and $f(2) = 3 > 0$, then there exists a root in $(0.4, 2)$.
Lastly, since f is continuous on $[-2, 0]$, $f(-2) = -1 < 0$ and $f(0) = 1 > 0$, then there exists a root in $(-2, 0)$.

The above shows that f has at least three distinct real roots. To show that there are only three distinct real roots, consider $f'(x) = 3\left(x^2 - 1\right)$ so f is strictly increasing for $x > 1$ and strictly decreasing for $x < -1$.

(ii) Note that

$$fg(x) = f\left(\frac{1}{1-x}\right) = \left(\frac{1}{1-x}\right)^3 - 3\left(\frac{1}{1-x}\right) + 1 = -\frac{1 - 3x + x^3}{(1-x)^3}$$

so $g(\alpha), g(\beta)$ and $g(\gamma)$ are the roots of f. From **(i)**, we know that $\alpha \in (-2,0)$, $\beta \in (0,0.4)$ and $\gamma \in (0.4,2)$. We have $g(\gamma) < 0$, which implies that $g(\gamma) = \alpha$. Suppose on the contrary that $g(\beta) = \beta$. Then,

$$\frac{1}{1-\beta} = \beta.$$

That is, $\beta^2 - \beta + 1 = 0$. However, the roots of this equation are not real, which is a contradiction. As such, $g(\beta) = \gamma$, leaving us with $g(\alpha) = \beta$.

(iii) Write $h(x) = ax^2 + bx + c$, where $a,b,c \in \mathbb{R}$ and $a \neq 0$. Then for $x = \alpha, \beta, \gamma$, we have

$$\begin{aligned} ax^2 + bx + c &= \frac{1}{1-x} \\ ax^2(1-x) + bx(1-x) + c(1-x) - 1 &= 0 \\ ax^2 - ax^3 + bx - bx^2 + c - cx - 1 &= 0 \\ -ax^3 + (a-b)x^2 + (b-c)x + c - 1 &= 0 \end{aligned}$$

Comparing the last line with $f(x)$, we see that $a = -1$, $b = -1$ and $c = 2$. So, $h(x) = -x^2 - x + 2$.

Remark for Question 4: This question is the same as Question 6 of the 2017 specimen paper.

Question 5

(a) The derangements are

$$2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321.$$

(b) For an ordering of the numbers 1 to n, let A_n denote the event that the number i is in position i. We wish to find

$$\left| \bigcap_{i=1}^{n} A_i' \right|.$$

By de Morgan's law, the above is equal to

$$n! - \left| \bigcup_{i=1}^{n} A_i \right|.$$

By the principle of inclusion and exclusion,

$$\begin{aligned} \left| \bigcup_{i=1}^{n} A_i \right| &= \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n| \\ &= n(n-1)! - \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^{n+2} \end{aligned}$$

So,

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! + \ldots + (-1)^{n+1} \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + (-1)^n \frac{1}{n!}\right) \end{aligned}$$

(c) As

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

then

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

So,

$$\begin{aligned} \left| D_n - \frac{n!}{e} \right| &= n! \left| \sum_{k=0}^{n} \frac{(-1)^k}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \right| \\ &= n! \left| \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right| \\ &= n! \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \\ &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \ldots \\ &= \frac{1}{n+1}\left[1 - \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} - \frac{1}{(n+2)(n+3)(n+4)} + \ldots\right] \end{aligned}$$

As

$$\frac{1}{n+2} - \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \ldots \in (0,1),$$

it follows that

$$\left| D_n - \frac{n!}{e} \right| < \frac{1}{1+n}.$$

For the second part, since $n \geq 1$, then $\left|D_n - \frac{n!}{e}\right| < 1$ and the result follows.

(d) We need to show that

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}.$$

This is true because

$$\lim_{n\to\infty} \frac{D_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{e}.$$

Remark for Question 5: For **(a)**, one sees that there are 9 derangements for the case when $n = 4$. One can verify this by using the formula given in **(b)**.

Question 6

(a) Consider the following figure. The sum of the areas of the rectangles is

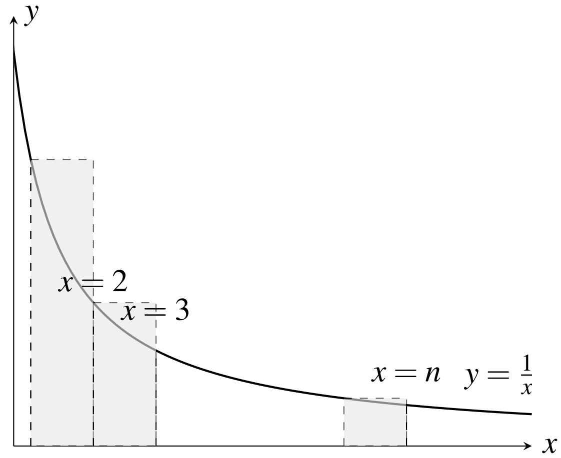
$$\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n-1}=H_{n-1},$$

whereas the area under the curve $y=\frac{1}{x}$ from $x=1$ to $x=n$ is

$$\int_1^n \frac{1}{x}\,dx=\ln n.$$

As such, $\ln n<H_{n-1}$. Adding $\frac{1}{n}$ to both sides and recognising the $H_{n-1}+\frac{1}{n}=H_n$, we obtain

$$\frac{1}{n}+\ln n<H_n.$$

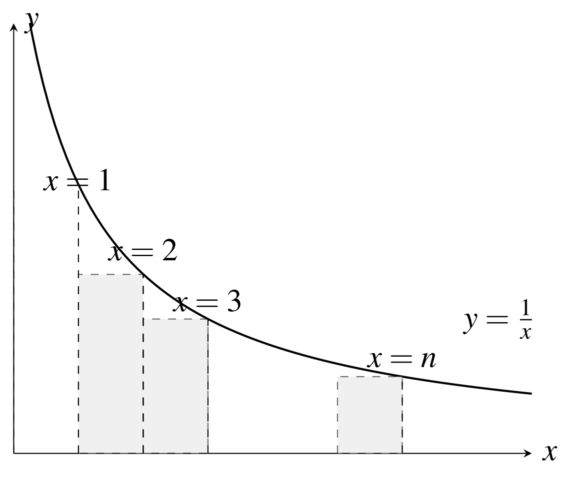


Next, consider the following figure. The sum of the areas of the rectangles is

$$\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}=H_n-1,$$

whereas the area under the curve $y=\frac{1}{x}$ from $x=1$ to $x=n$ is $\ln n$. As such, $H_n-1<\ln n$. Adding 1 to both sides, it follows that

$$H_n<1+\ln n.$$



(b) We have

$$\begin{aligned}\lim_{n\to\infty} H_n &> \lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\ln n\\ &= 1+\lim_{n\to\infty}\ln n\end{aligned}$$

Since $\ln n$ diverges to infinity, then the harmonic series diverges too.

Next, $H_{1\ 000\ 000} < 1+\ln(1\ 000\ 000) = 14.815 < 15$.

(c) As $p_n = n!H_n$, it suffices to show that

$$n!H_n = n(n-1)!H_{n-1}+(n-1)!.$$

Starting with the RHS, we have

$$\begin{aligned}n(n-1)!H_{n-1}+(n-1)! &= (n-1)!(nH_{n-1}+1)\\ &= n!\left(\frac{nH_{n-1}+1}{n}\right)\\ &= n!\left(H_{n-1}+\frac{1}{n}\right)\\ &= n!H_n\end{aligned}$$

(d) **(i)** $\begin{bmatrix} n \\ k \end{bmatrix}$ counts the number of ways for n distinct people to sit around 1 circular table. This is equivalent to the number of permutations of n distinct objects on a circle, which is $(n-1)!$.

(ii) Consider a person out of the n, say α. We have the following two cases:

- **Case 1:** If α is alone, then there are $\begin{bmatrix} n \\ k-1 \end{bmatrix}$ ways to distribute the remaining n people around $k-1$ tables such that no table is empty.
- **Case 2:** If α is seated with other people, we first let α sit around some arbitrary table in n ways. Then, distribute the remaining n people around k tables such that the other tables are non-empty in $\begin{bmatrix} n \\ k \end{bmatrix}$ ways.

By the addition principle, it follows that $\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n\begin{bmatrix} n \\ k \end{bmatrix}$.

(e) From **(d)**, we have

$$\begin{aligned}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \begin{bmatrix} n \\ 1 \end{bmatrix} + n\begin{bmatrix} n \\ 2 \end{bmatrix}\\ &= (n-1)!+n\begin{bmatrix} n \\ 2 \end{bmatrix}\\ \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} &= \frac{1}{n}+\frac{1}{(n-1)!}\begin{bmatrix} n \\ 2 \end{bmatrix}\end{aligned}$$

Let $f(k)=\dfrac{1}{(k-1)!}\begin{bmatrix}k\\2\end{bmatrix}$. Then, $f(k)$ satisfies $f(k+1)-f(k)=\frac{1}{k}$.
Summing both sides, we have

$$\sum_{k=2}^{n-1}[f(k+1)-f(k)]=\sum_{k=2}^{n-1}\frac{1}{k},$$

and it follows by the method of difference that $f(n)-f(2)=H_{n-1}-1$. Since $f(2)=1$, then $f(n)=H_{n-1}$.
Therefore, $f(n+1)=H_n$ and the result follows.

(f) We have,

$$\begin{aligned}2^kM(n)H_n&=\frac{2^kM(n)}{1}+\frac{2^kM(n)}{2}+\frac{2^kM(n)}{3}+\ldots+\frac{2^kM(n)}{2^k}+\ldots+\frac{2^kM(n)}{n}\\&=2^k\left[\frac{M(n)}{1}\right]+2^k\left[\frac{M(n)}{2}\right]+2^k\left[\frac{M(n)}{3}\right]+\ldots+M(n)+\ldots+2^k\left[\frac{M(n)}{n}\right]\end{aligned}$$

- **Case 1:** Suppose n is odd. Then, $M(n)=1\cdot 3\cdot 5\cdot\ldots\cdot n$.
- **Case 2:** Suppose n is even. Then, $M(n)=1\cdot 3\cdot 5\ldots\cdot(n-1)$. Also, 2^k and n have a common factor of 2, which shows that $\frac{M(n)}{n}$ is an integer.

In each case, since $2^k\left[\dfrac{M(n)}{i}\right]$ is even for all i except when $i=2^k$, the result follows.

(g) Suppose on the contrary that there exists $\beta\in\mathbb{Z}$ such that $H_n=\beta$.
Then, $2^kM(n)H_n=2^kM(n)\beta$. From **(f)**, the LHS is odd, but the RHS is even.
This is a contradiction so no such $\beta\in\mathbb{Z}$ exists.

Remark for Question 6: In **(d)**, $\begin{bmatrix}n\\k\end{bmatrix}$ is related to the Stirling numbers of the first kind.